

A Numerical Solution Based On Operational Matrix of Differentiation of Shifted Second Kind Chebyshev Wavelets for a Stefan Problem

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Abstract—In this study, one dimensional phase change problem (a Stefan problem) is considered and a numerical solution of this problem is discussed. First, we use similarity transformation to convert the governing equations into ordinary differential equations with its boundary conditions. The solutions of ordinary differential equation with the associated boundary conditions and interface condition (Stefan condition) are obtained by using a numerical approach based on operational matrix of differentiation of shifted second kind Chebyshev wavelets. The obtained results are compared with existing exact solution which is sufficiently accurate.

Keywords—Operational matrix of differentiation, Similarity transformation, Shifted second kind Chebyshev wavelets, Stefan problem.

I. INTRODUCTION

MOVING boundary problems (Stefan problems) arise in many important areas of science and engineering. These problems have wide applications in industries, food technology, vehicle design, growing crystals for semiconductors, image development in electro photography, cryosurgery, plasma physics, geophysics, etc. The solutions of moving boundary problems have been of special interest due to the inherent difficulties associated with its nonlinear nature and presence of moving boundary/boundaries. The history and various solutions (exact, approximate and numerical) are well covered in [1]-[3]. Some recent approach to moving boundary problem can also be seen in [4], [5].

Recently, numerical algorithms based on wavelets have drawn a great attention for solving linear and non-linear differential equations. These numerical algorithms are remarkable due to its simplicity and accuracy. Moreover, rate of convergence of numerical techniques based on finite difference and finite element method are algebraic. But, Rate of convergence of numerical methods based on wavelets is exponential in simple geometry [15]. Several differential and integrals equations are solved by using Legendre wavelets [6], [7] and Chebyshev wavelets [8]-[17].

In this study, a numerical solution of a one-dimensional phase change problem is obtained. The solution of the problem is based on similarity transformation and Operational

matrix of differentiation of shifted Chebyshev polynomial of second kind wavelets. The comparisons between obtained numerical results and existing analytical solutions are also shown through figures and table.

II. SECOND KIND CHEBYSHEV POLYNOMIALS AND THEIR SHIFTED FORM

Chebyshev polynomials of second kind are defined on $[-1, 1]$ as:

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta. \quad (1)$$

These polynomials are orthogonal on $[-1, 1]$ with respect to weight function $\sqrt{1-x^2}$, i.e.

$$\int_{-1}^1 \sqrt{1-x^2} U_n(x) U_m(x) dx = \begin{cases} 0, & n \neq m, \\ \frac{\pi}{2}, & n = m. \end{cases} \quad (2)$$

Chebyshev polynomials of second kind can also be found from following Rodrigues formula:

$$U_n(x) = \frac{(-2)^n (n+1)!}{(2n+1)! \sqrt{1-x^2}} D^n \left[(1-x^2)^{n+\frac{1}{2}} \right]. \quad (3)$$

The shifted second kind Chebyshev polynomials are defined on $[0, 1]$ by $U_n^*(x) = U_n(2x-1)$. These polynomials are orthogonal on $[0, 1]$ with respect to weight function $\sqrt{x-x^2}$; that is,

$$\int_0^1 \sqrt{x-x^2} U_n^*(x) U_m^*(x) dx = \begin{cases} 0, & n \neq m, \\ \frac{\pi}{8}, & n = m. \end{cases} \quad (4)$$

III. OPERATIONAL MATRIX OF DERIVATIVES FOR SHIFTED SECOND KIND CHEBYSHEV WAVELETS

Wavelets include a family of functions which are constructed from dilation and translation of single function. This single function is called the mother wavelet. When the dilation parameter and the translation parameter vary continuously, then the following family of continuous wavelets can be found (see [15]-[17]):

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$$\psi_{a,b}(t) = |a|^{-1/2} \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathfrak{R}, a \neq 0, \quad (5)$$

where a and b are the dilation parameter and the translation parameter, respectively.

Second kind Chebyshev wavelets $\psi_{nm}(t)$ are defined on the interval $[0, 1]$ as:

$$\psi_{nm}(t) = \begin{cases} \frac{2^{(k+3)/2}}{\sqrt{\pi}} U_m^*(2^k t - n), & t \in \left[\frac{n}{2^k}, \frac{n+1}{2^k}\right], \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

where $m = 0, 1, 2, \dots, M, n = 0, 1, 2, \dots, 2^k - 1$.

Clearly, second kind Chebyshev wavelets have four arguments k, n, m (order of second kind Chebyshev polynomials), and t (normalized time). In this paper, the following properties of second kind Chebyshev wavelets (given in [15]-[17]) are used:

a. In terms of shifted second kind Chebyshev wavelets, a function $f(t)$ defined over $[0, 1]$ may be expressed as:

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{nm} \psi_{nm}(t), \quad (7)$$

where

$$C_{nm} = \langle f(t), \psi_{nm}(t) \rangle = \int_0^1 \sqrt{1-t^2} f(t) \psi_{nm}(t) dt. \quad (8)$$

If the infinite series of $f(t)$ is truncated, then (7) becomes

$$f(t) \approx \sum_{n=0}^{2^k-1} \sum_{m=0}^M C_{nm} \psi_{nm}(t) = C \psi(t), \quad (9)$$

where C is $1 \times 2^k(M+1)$ and $\psi(t)$ is $2^k(M+1) \times 1$ matrices which are defined as:

$$C = [c_{0,0}, c_{0,1}, \dots, c_{0,M}, \dots, c_{2^k-1,0}, c_{2^k-1,1}, \dots, c_{2^k-1,M}], \quad (10)$$

$$\psi(t) = [\psi_{0,0}, \psi_{0,1}, \dots, \psi_{0,M}, \dots, \psi_{2^k-1,0}, \psi_{2^k-1,1}, \dots, \psi_{2^k-1,M}]^T. \quad (11)$$

b. If $\psi(t)$ is the second kind Chebyshev wavelets vector then the first derivative of $\psi(t)$ can be defined as:

$$\frac{d\psi(t)}{dt} = D\psi(t), \quad (12)$$

where D is $2^k(M+1)$ square operational matrix of derivative of shifted second kind Chebyshev wavelets (see proof in [15]). The structure of D is given as:

$$D = \begin{bmatrix} G & 0 & \dots & 0 \\ 0 & G & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & G \end{bmatrix} \quad (13)$$

where G is an $(M+1)$ square matrix whose (r, s) th element is given by

$$G_{r,s} = \begin{cases} 2^{k+2} s, & r \geq 2, r > s, (r+s) \text{ odd}, \\ 0, & \text{otherwise.} \end{cases} \quad (14)$$

c. The operational matrix for the n th derivative of second kind Chebyshev wavelets vector can be derived as:

$$\frac{d^n \psi(t)}{dt^n} = D^n \psi(t), \quad n = 1, 2, \dots, \quad (15)$$

where D^n is the n th power of matrix D .

IV. MATHEMATICAL MODEL

In this section, a problem of solidification of liquid is considered in semi-infinite domain ($0 \leq y < \infty$). Initially, the liquid is assumed at its fusion temperature θ_f . At time $\tau = 0$, a temperature θ_0 ($\theta_0 < \theta_f$) is imposed at $y = 0$. As time proceeds, solidification process starts and the governing model of this process [1], [2] are formulated as:

$$\frac{\partial \theta(y, \tau)}{\partial \tau} = \alpha \frac{\partial^2 \theta(y, \tau)}{\partial y^2}, \quad 0 < y < S(\tau) \quad (16)$$

$$\theta(0, \tau) = \theta_0, \quad \theta(S(\tau), \tau) = \theta_f, \quad (17)$$

$$\rho h \frac{dS(\tau)}{d\tau} = -k \frac{\partial \theta(y, \tau)}{\partial y} \text{ at } y = S(\tau), \quad (18)$$

$$S(0) = 0, \quad (19)$$

where θ is the temperature distribution in solid region, α is the solid diffusivity, ρ is the liquid density, h is the latent heat, k is the thermal conductivity of solid, τ is the time and $S(\tau)$ is the position of moving interface.

Now, we consider the following dimensionless variables [2]:

$$\begin{aligned} x &= \frac{y}{l}, & t &= \frac{\alpha \tau}{l^2}, & T &= \frac{\theta_f - \theta}{\Delta \theta_{ref}}, \\ s &= \frac{S}{l}, & Ste &= \frac{L}{c \Delta \theta_{ref}} \end{aligned} \quad (20)$$

where l is the characteristic length (any convenient length), $\Delta \theta_{ref} (= \theta_f - \theta_0)$ is a reference temperature and Ste is the Stefan number.

Introducing above dimensionless variables into (16)-(19), one can get the following equations in dimensionless form:

$$\frac{\partial T(x, t)}{\partial t} = \frac{\partial^2 T(x, t)}{\partial x^2}, \quad 0 < x < s(t), \quad (21)$$

$$T(0, t) = 1, \quad T(s(t), t) = 0, \quad (22)$$

$$\frac{ds(t)}{dt} = -Ste \frac{\partial T(x, t)}{\partial x} \text{ at } x = s(t), \quad (23)$$

$$s(0) = 0. \quad (24)$$

V. SOLUTION OF THE PROBLEM

First, we consider the following similarity transformation as given in [1], [2]:

$$T(x, t) = \eta(\xi), \text{ with } \xi = \frac{x}{\sqrt{t}} \quad (25)$$

and

$$s(t) = \sqrt{2\lambda t}, \quad (26)$$

where λ is a positive constant.

Under the above transformation, (21)-(24) become

$$\eta''(\xi) + \frac{\xi}{2} \eta'(\xi) = 0, \quad (27)$$

$$\eta(0) = 1, \quad \eta(\sqrt{2\lambda}) = 0, \quad (28)$$

and

$$\eta'(\sqrt{2\lambda}) = -Ste\sqrt{\frac{\lambda}{2}}. \quad (29)$$

Now, approximating $\eta(\xi)$ and ξ in terms of the second kind Chebyshev wavelets [15] as:

$$\eta(\xi) = \sum_{n=0}^{2^k-1} \sum_{m=0}^M C_{nm} \psi_{nm}(\xi) = C\psi(\xi), \quad (30)$$

and

$$\xi = \sum_{n=0}^{2^k-1} \sum_{m=0}^M f_{nm} \psi_{nm}(\xi) = F\psi(\xi), \quad (31)$$

where

$$C = [c_{0,0}, c_{0,1}, \dots, c_{0,M}, \dots, c_{2^k-1,0}, c_{2^k-1,1}, \dots, c_{2^k-1,M}],$$

$$\psi(t) = [\psi_{0,0}, \psi_{0,1}, \dots, \psi_{0,M}, \dots, \psi_{2^k-1,0}, \psi_{2^k-1,1}, \dots, \psi_{2^k-1,M}]^T,$$

and

$$F = [f_{0,0}, f_{0,1}, \dots, f_{0,M}, \dots, f_{2^k-1,0}, f_{2^k-1,1}, \dots, f_{2^k-1,M}].$$

From (15), we have

$$\eta'(\xi) = CD\psi(\xi), \quad \eta''(\xi) = CD^2\psi(\xi). \quad (32)$$

Substituting (30)-(32) into (27), we get

$$CD^2\psi(\xi) + \frac{1}{2}F\psi(\xi)CD\psi(\xi) = 0. \quad (33)$$

Equations (28) and (29) give

$$C\psi(0) = 1, \quad C\psi(\sqrt{2\lambda}) = 0, \quad (34)$$

and

$$CD\psi(\sqrt{2\lambda}) = -Ste\sqrt{\lambda/2}, \quad (35)$$

respectively.

As given in [15]-[16], (33) generates $(2^k(M+1)-2)$ equations at the first $(2^k(M+1)-2)$ roots of $U_{2^k(M+1)}^*(\xi)$ and (34)-(35) will also produce three more equations in terms of $(2^k(M+1)+1)$ unknowns ($2^k(M+1)$ constants and one λ). These equations can be solved by any appropriate numerical method and after getting these unknowns one can calculate temperature distribution in the domain and position of interface.

VI. NUMERICAL RESULTS AND DISCUSSION

In this section, all numerical computations have been done by considering $M=2, k=0$ and results are presented through figures. Hence, the following matrices D, D^2, C, ψ and F are considered in the calculations:

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{bmatrix}, \quad D^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 32 & 0 & 0 \end{bmatrix}, \quad C = \sqrt{\frac{\pi}{2}} [c_0, c_1, c_2],$$

$$\psi(\xi) = \sqrt{\frac{2}{\pi}} \begin{bmatrix} 2 \\ 8\xi - 4 \\ 32\xi^2 - 32\xi + 6 \end{bmatrix}, \quad \text{and } F = \sqrt{\frac{\pi}{2}} \left[\frac{1}{2\sqrt{2\pi}}, \frac{1}{4\sqrt{2\pi}}, 0 \right].$$

TABLE I
 COMPARISONS BETWEEN EXACT SOLUTIONS AND NUMERICAL SOLUTIONS

Ste	Time (t)	Exact value of s(t)	Approximate value of s(t)	Absolute Error
1.0	0.0	0.0	0.0	0.0
	0.05	0.34391	0.32771	0.0162
	0.10	0.48636	0.46346	0.0229
	0.15	0.59566	0.56762	0.0280
	0.20	0.68782	0.65543	0.0324
	0.25	0.76900	0.73279	0.0362
	0.30	0.84239	0.80273	0.0396
	0.35	0.90989	0.86705	0.0428
	0.40	0.97272	0.92692	0.0458
	5.0	0.0	0.0	0.0
0.5		0.26558	0.25015	0.0154
1.0		0.37560	0.35377	0.0218
1.5		0.46000	0.43328	0.0267
2.0		0.53118	0.50031	0.0308
2.5		0.59387	0.55936	0.0345
3.0		0.65055	0.61275	0.0378
3.5		0.70268	0.66184	0.0408
4.0		0.75120	0.70754	0.0436
10.0		0.0	0.0	0.0
	2.0	0.27379	0.26001	0.0137
	4.0	0.38720	0.36771	0.0195
	6.0	0.47422	0.45035	0.0238
	8.0	0.54758	0.52003	0.0275
	10.0	0.61222	0.58141	0.0308
	12.0	0.67065	0.63690	0.0337
	14.0	0.72438	0.68793	0.0364
	16.0	0.77440	0.73543	0.0389

Table I shows the comparison between exact solution and obtained numerical solution for different value of Stefan numbers. Figs. 1-3 represents accuracy of our results with the existing exact solution (given in [1], [2]) for the trajectories of interface at different value of Stefan numbers ($Ste = 1.0, 5.0, 10.0$). It is seen from the table and figures that the obtained numerical solution is sufficiently closed to the exact solution.

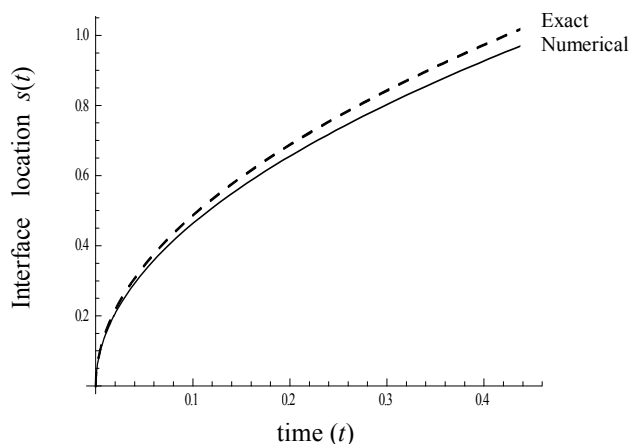


Fig. 1 Plot of $s(t)$ vs t at $Ste = 1.0$

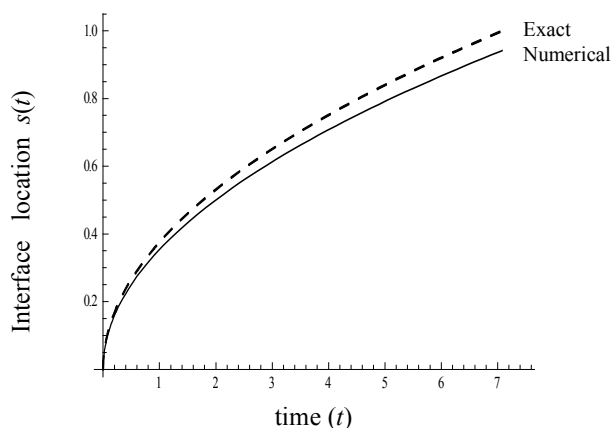


Fig. 2 Plot of $s(t)$ vs t at $Ste = 5.0$

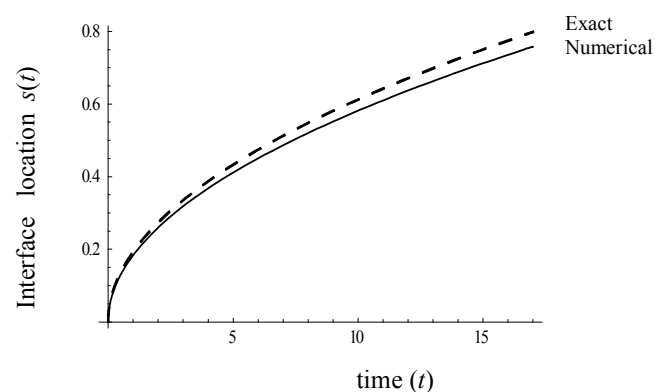


Fig. 3 Plot of $s(t)$ vs t at $Ste = 10.0$

VII. CONCLUSION

In this paper, a Stefan problem governing the freezing process is considered and its numerical solution is obtained by using operational matrix of differentiation of shifted second kind Chebyshev wavelets. It is found that this approach is a simple technique and the tactic is computer intensive. Moreover, the obtained results are sufficiently accurate. The authors believe that the accuracy of the approach can be increased by considering/taking higher values of M and k .

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