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Alternative Robust Estimators for the Shape Parameters of the Burr XII Distribution

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***Abstract*—**In general, classical methods such as maximum likelihood (ML) and least squares (LS) estimation methods are used to estimate the shape parameters of the Burr XII distribution. However, these estimators are very sensitive to the outliers. To overcome this problem we propose alternative robust estimators based on the *M*-estimation method for the shape parameters of the Burr XII distribution. We provide a small simulation study and a real data example to illustrate the performance of the proposed estimators over the *ML* and the *LS* estimators. The simulation results show that the proposed robust estimators generally outperform the classical estimators in terms of bias and root mean square errors when there are outliers in data.

***Keywords*—**Burr XII distribution, robust estimator, *M*-estimator, maximum likelihood, least squares.

I. Introduction

THE Burr XII distribution was first introduced by [1]. This distribution is also known as “Singh-Maddala” distribution. It has been widely used as a model in areas such as business [2], economics [3], engineering [4], finance [4], hydrology [5], mineralogy [6], medical research [7] and reliability analysis [7]–[10]. Its properties were studied by [11] and [12]. Details on the connection between the Burr XII distribution and other continuous distributions were given by [13], [14].

The shape parameters of Burr XII distribution have been estimated by using the *ML*, *LS* and the maximum product of spacing estimation methods. The *ML* estimation method was used by [7]. The *ML*, *LS* and the maximum product of spacing estimation methods were compared by [15] in presence and in absence of outliers. Also, the *ML* and the maximum product of spacing estimation methods were compared by [16].

The estimators obtained from the classical methods perform well when there are no outliers in the data. However, it is known that these estimators are very sensitive to the outliers. Therefore, robust estimation methods should be used to estimate the shape parameters of the Burr XII distribution. The robust estimators will perform as good as the classical estimators when there are no outliers in the data and will be less influenced by the outliers if there are some potential outliers. For this reason, the robust regression estimation method was used by [17] to estimate the shape parameters of the Burr XII distribution for complete and multiply-censored data with outliers. The optimal *B*-robust (*OBR*) estimation method is proposed by [18] to obtain robust estimators for the shape parameters of this distribution.

In this paper, an alternative robust estimation method based on the *M*-estimation method is proposed to estimate the shape parameters of the Burr XII distribution. The proposed method is based on minimizing a robust objective function instead of *LS* objective function.

The rest of the paper is organized as follows. In Section II, we give some properties of the Burr XII distribution. In Section III, we summarize the *ML* and the *LS* estimation methods and describe the proposed robust estimation method for the shape parameters of the Burr XII distribution. In Sections IV and V, we give a small simulation study and a real data example to compare the performances of the proposed estimators with the *ML* and the *LS* estimators. The paper is finalized with a conclusion section.

II. The Burr XII Distribution

The cumulative density function (cdf) and probability density function (pdf) for a two parameter Burr XII distribution are given by:

$$F(x) = 1 - \frac{1}{(1+x^c)^k}, x \geq 0, c > 0, k > 0, \tag{1}$$

$$f(x) = ck\frac{x^{c-1}}{(1+x^c)^{k+1}}, x \geq 0, c > 0, k > 0, \tag{2}$$

where c and k are shape parameters. If a random variable X has a Burr XII distribution, then the r th moment of X is

$$E(X^r) = k\,\Gamma(k - r/c)\Gamma(r/c+1)/\Gamma(k+1), r < ck \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function.

To have the fourth moment ck should be greater than 4. By using (3), expected value, variance, skewness $(\sqrt{\beta_1})$ and kurtosis (β_2) for the Burr XII distribution can be obtained as

$$E(X) = kB\left(\frac{1}{c+1}, \frac{k-1}{c}\right), \tag{4}$$

$$Var(X) = kB\left(\frac{2}{c+1}, \frac{k-2}{c}\right) - \left(kB\left(\frac{1}{c+1}, \frac{k-1}{c}\right)\right)^2, \tag{5}$$

$$\sqrt{\beta_1} = \frac{\Gamma^2(k)\lambda_3 - 3\Gamma(k)\lambda_2\lambda_1 + 2\lambda_1^3}{\left(\Gamma(k)\lambda_2 - \lambda_1^2\right)^{3/2}}, \tag{6}$$

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$$\beta_2 = \frac{\Gamma^3(k)\lambda_4 - 4\Gamma^2(k)\lambda_3\lambda_1 + 6\Gamma(k)\lambda_1^2 - 3\lambda_1^4}{\left(\Gamma(k)\lambda_2 - \lambda_1^2\right)^{3/2}}, \quad (7)$$

where $B(\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ is the beta function and $\lambda_j = \Gamma(j/c+1)\Gamma(k-j/c), j=1,2,3,4.$ Fig. 1 shows some example of the Burr XII density function for some values of c and k. We can see from the pdf plots that the shape parameters c and k control the peakedness, skewness and the tail thickness of the Burr XII distribution. We can observe that if $c>1$, the pdf is unimodal with the mode at $x = \left((c-1)/(kc+1)^{1/c}\right)$, and if $c \le 1$, the pdf has L-shaped.

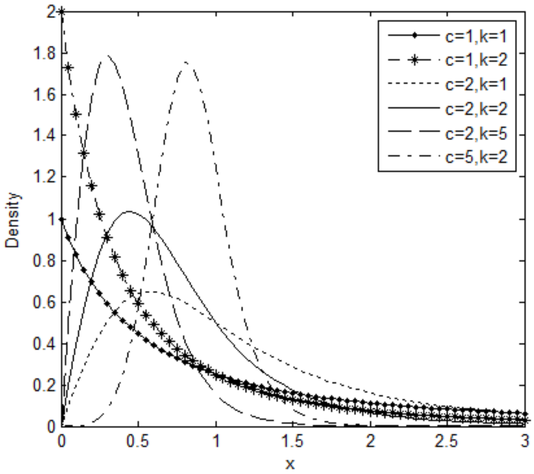

Fig. 1 Examples of the Burr XII density function for different values of c and k

III. Parameters Estimation

In this section, we will first review the *ML* and the *LS* estimation methods to estimate the shape parameters of the Burr XII distribution. We will then introduce the robust estimators based on the *M*-estimation method given in [19].

A. ML Estimation Method

Let $X = (x_1, x_2, ..., x_n)$ be a random sample from a Burr XII distribution and assume that the shape parameters c and k are unknown. The *ML* estimators for c and k can be found by maximizing the following log-likelihood function

$$\log L(c,k) = n\log c + n\log k + (c-1)\sum_{i=1}^{n}\log x_i - (k+1)\sum_{i=1}^{n}\log\left(x_i^c+1\right). \quad (8)$$

After taking the derivative of $\log L(c,k)$ with respect to c and k and setting to zero, the following equations are obtained

$$\frac{n}{\hat{k}} - \sum_{i=1}^{n}\log\left(x_i^{\hat{c}}+1\right) = 0, \quad (9)$$

$$\frac{n}{\hat{c}} + \sum_{i=1}^{n}\log x_i - \left(\hat{k}+1\right)\sum_{i=1}^{n}\frac{x_i^{\hat{c}}}{1+x_i^{\hat{c}}}\log x_i = 0. \quad (10)$$

Since the above equations cannot be solved analytically, numerical methods should be used to obtain the *ML* estimates.

B. LS Estimation Method

The shape parameters c and k are estimated by [15] using the *LS* estimation method. The *LS* estimators for c and k can be obtained by minimizing the following objective function with respect to c and k

$$S = \sum_{i=1}^{n}\left(y_i - \log k - \log\log\left(1+x_i^c\right)\right)^2, \quad (11)$$

where $y_i = \log\log\left(\frac{1}{1-F(x)}\right) = \log k + \log\log\left(1+x_i^c\right).$ Since $F(x)$ is unknown the following approximation can be used to determine the values of y_i

$$E\left(F\left(x_{(i)}\right)\right) = \frac{i-0.5}{n}, i = 1,2,...,n, \quad (12)$$

where $x_{(i)}$ are the order statistics and $F\left(x_{(i)}\right)$ is uniformly distributed on $(0,1)$. The following equations should be solved to obtain the *LS* estimators for c and k

$$\log\hat{k} = \bar{y} - \frac{1}{n}\sum_{i=1}^{n}\log\log\left(1+x_i^{\hat{c}}\right), \quad (13)$$

$$\sum_{i=1}^{n}\frac{x_i^{\hat{c}}\log x_i\left(y_i - \log\log\left(1+x_i^{\hat{c}}\right) - \log\hat{k}\right)}{\left(1+x_i^{\hat{c}}\right)\log\left(1+x_i^{\hat{c}}\right)} = 0. \quad (14)$$

Similarly, since these equations cannot also be solved explicitly, numerical methods should be used to obtain the *LS* estimates.

C. Robust Estimation Method

Since the *LS* estimators are sensitive to the outliers in data, robust estimators are proposed by [17] and [18] to estimate the shape parameters of Burr XII distribution. In those papers, the robust regression and the *OBR* estimation methods are used to obtain robust estimators for c and k. In the present paper, we propose alternative robust estimators for the shape parameters of the Burr XII distribution by minimizing the following objective function with respect to the parameters c and k

$$Q(c,k) = \sum_{i=1}^{n}\rho\left(y_i - \log k - \log\log\left(1+x_i^c\right)\right). \quad (15)$$

We will use two different ρ functions. One of them is the Huber's ρ function

$$\rho(x)=\begin{cases} x^2, |x|\le b \\ 2b|x|-b^2, |x|>b \end{cases} \tag{16}$$

with the derivative

$$\psi(x)=\begin{cases} x, |x|\le b \\ sign(x)b, |x|>b \end{cases}. \tag{17}$$

The other is the Tukey's ρ function

$$\rho(x)=\begin{cases} 1-\left(1-(x/b)^2\right)^3, |x|\le b \\ 1, |x|>b \end{cases} \tag{18}$$

with the derivative

$$\psi(x)=x\left[1-\left(\frac{x}{b}\right)^2\right]^2 I\left(|x|\le b\right). \tag{19}$$

Taking the derivatives of the objective function given in (15) with respect to c and k and setting to zero give the following equations

$$\frac{\partial Q}{\partial k}=\sum_{j=1}^{n}\rho'\left(y_i-\log k-\log\log\left(1+x_i^c\right)\right)\frac{1}{k}=0, \tag{20}$$

$$\frac{\partial Q}{\partial c}=\sum_{i=1}^{n}\frac{\rho'\left(y_i-\log k-\log\log\left(1+x_i^c\right)\right)}{\left(1+x_i^c\right)\log\left(1+x_i^c\right)}x_i^c\log x_i=0. \tag{21}$$

We can further simplify these equations as

$$\log\hat{k}=\frac{\sum_{i=1}^{n}w_i y_i}{\sum_{i=1}^{n}w_i}-\frac{\sum_{i=1}^{n}w_i\log\log\left(1+x_i^c\right)}{\sum_{i=1}^{n}w_i}, \tag{22}$$

$$\sum_{i=1}^{n}\frac{w_i\left(y_i-\log\hat{k}-\log\log\left(1+x_i^{\hat{c}}\right)\right)}{\left(1+x_i^{\hat{c}}\right)\log\left(1+x_i^{\hat{c}}\right)}x_i^{\hat{c}}\log x_i=0, \tag{23}$$

where $w_i, i=1,...,n$ are the weights. The weights for the Huber's ρ function are obtained as

$$w_i=\min\left\{1,\frac{b}{\left|y_i-\log k-\log\log\left(1+x_i^c\right)\right|}\right\}. \tag{24}$$

If $\left|y_i-\log k-\log\log\left(1+x_i^c\right)\right|\le b,$ the weight will be 1, otherwise it will be $\frac{b}{\left|y_i-\log k-\log\log\left(1+x_i^c\right)\right|}$. The weights for the Tukey's ρ function are

$$w_i=\left(1-\left(\frac{y_i-\log k-\log\log\left(1+x_i^c\right)}{b}\right)^2\right)^2 I\left(\left|y_i-\log k-\log\log\left(1+x_i^c\right)\right|\le b\right). \tag{25}$$

The weight will be $\left(1-\left(\frac{y_i-\log k-\log\log\left(1+x_i^c\right)}{b}\right)^2\right)^2$ if $\left|y_i-\log k-\log\log\left(1+x_i^c\right)\right|\le b,$ otherwise it will be 0. We can observe that the weights eliminate the effect of outliers on the estimators.

Since we cannot get explicit solutions of these equations, they should also be solved numerically to obtain the robust estimates for the parameters c and k.

IV. Simulation Study

In this section, we provide a small simulation study to compare the performances of the *ML, LS* and the robust estimators with and without outliers. The data were generated from the Burr XII distribution for several different values of c and k. To evaluate the performance of the estimators, bias and root mean square errors (RMSE)

$$bias\left(\hat{c}\right)=\overline{c}-c, bias\left(\hat{k}\right)=\overline{k}-k, \tag{26}$$

$$RMSE(\hat{c})=\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{c}_i-c\right)^2}, \tag{27}$$

$$RMSE(\hat{k})=\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{k}_i-k\right)^2}. \tag{28}$$

They were calculated using 1000 replications for each sample sizes and parameter values, where $\overline{c}=1/N\sum_{i=1}^{N}\hat{c}_i, \overline{k}=1/N\sum_{i=1}^{N}\hat{k}_i$ and $N=1000$. Note that we have tried several different number of replications, but the results were very similar to the case $N=1000$. That's why we take the number of replications as 1000.

In the simulation study we set $n=20,40$ and 80. The case $n=20,40$ and 80 represent the small, moderate and the large sample sizes, respectively. We also take $(c,k)=(1,1),(1,2),(2,1),(2,2),(2,5),(5,2)$ to compare our results with the results given in literature (see [9], [15]). For each simulation configuration, the biases and the *RMSE*s were obtained with and without outliers. Tuning constants for Tukey's ρ and Huber's ρ functions are taken as 4 and 1.345, respectively. The simulation study and real data example are conducted using MATLAB 2013a.

Simulation results are summarized in Tables I-VI. In these tables $bias(\hat{c}), bias(\hat{k}), RMSE(\hat{c})$ and $RMSE(\hat{k})$ are provided. The simulation results for the datasets without outliers are given in Tables I-III. We can observe from these tables that although all the estimation methods considered in this paper have very similar performance when there are no outliers, the simulation results obtained from Tukey's ρ function generally show superiority in terms of *RMSE* and bias. The absence of outliers in the data leads to an over-estimation of the parameters in most simulation conditions.

TABLE I
THE BIAS AND RMSE (PARENTHESES) FOR $n = 20$ WITHOUT OUTLIERS

n	(c,k)	*ML*	*LS*	Huber	Tukey
		Parameter (c)			
20	(1,1)	0.0699	0.0243	0.0417	0.0646
		(0.2453)	(0.2520)	(0.2478)	(0.2466)
	(1,2)	0.0633	0.0139	0.0377	0.0345
		(0.1990)	(0.2219)	(0.1562)	(0.1489)
	(2,1)	0.1517	0.0602	0.1339	0.1309
		(0.5169)	(0.5199)	(0.5385)	(0.5217)
	(2,2)	0.1065	0.0260	0.0751	0.0828
		(0.4067)	(0.4446)	(0.3087)	(0.3039)
	(2,5)	0.1300	0.0181	0.0373	0.0320
		(0.3991)	(0.4148)	(0.1323)	(0.1446)
	(5,2)	0.3861	0.1470	0.2140	0.2525
		(1.0494)	(1.1070)	(0.7472)	(0.7518)
		Parameter (k)			
20	(1,1)	0.0410	0.0296	0.0644	0.0613
		(0.2723)	(0.2655)	(0.3230)	(0.3153)
	(1,2)	0.1522	0.0745	0.1157	0.1112
		(0.6080)	(0.6081)	(0.6404)	(0.6297)
	(2,1)	0.0361	0.0274	0.0568	0.0536
		(0.2786)	(0.2738)	(0.3134)	(0.3095)
	(2,2)	0.1049	0.0370	0.1014	0.0953
		(0.5194)	(0.5076)	(0.6338)	(0.6187)
	(2,5)	0.9003	0.4097	0.2704	0.2555
		(2.4891)	(2.2440)	(1.4968)	(1.4882)
	(5,2)	0.1329	0.0552	0.0771	0.0724
		(0.5869)	(0.5750)	(0.6390)	(0.6245)

TABLE II
THE BIAS AND RMSE (PARENTHESES) FOR $n = 40$ WITHOUT OUTLIERS

n	(c,k)	*ML*	*LS*	Huber	Tukey
		Parameter (c)			
40	(1,1)	0.0384	0.0104	0.0336	0.0398
		(0.1660)	(0.1786)	(0.1789)	(0.1786)
	(1,2)	0.0335	0.0052	0.0241	0.0245
		(0.1367)	(0.1585)	(0.1163)	(0.1113)
	(2,1)	0.0760	0.0186	0.0659	0.0682
		(0.3212)	(0.3452)	(0.3298)	(0.3264)
	(2,2)	0.0749	0.0279	0.0756	0.0714
		(0.2693)	(0.3145)	(0.2125)	(0.2129)
	(2,5)	0.0641	0.0109	0.0335	0.0305
		(0.2603)	(0.3002)	(0.1087)	(0.1054)
	(5,2)	0.1214	-0.0127	0.0946	0.1168
		(0.6587)	(0.7533)	(0.5253)	(0.5371)
		Parameter (k)			
40	(1,1)	0.0092	0.0030	0.0212	0.0197
		(0.1766)	(0.1732)	(0.2082)	(0.2031)
	(1,2)	0.0530	0.0087	0.0391	0.0334
		(0.3528)	(0.3714)	(0.4283)	(0.4191)
	(2,1)	0.0158	0.0102	0.0259	0.0253
		(0.1843)	(0.1828)	(0.2116)	(0.2096)
	(2,2)	0.0767	0.0421	0.0505	0.0443
		(0.3799)	(0.3976)	(0.4362)	(0.4256)
	(2,5)	0.3515	0.1469	0.1074	0.1047
		(1.2625)	(1.3344)	(1.1094)	(1.0973)
	(5,2)	0.0699	0.0259	0.0729	0.0688
		(0.3666)	(0.3737)	(0.4409)	(0.4334)

TABLE III
THE BIAS AND RMSE (PARENTHESES) FOR $n = 80$ WITHOUT OUTLIERS

n	(c,k)	*ML*	*LS*	Huber	Tukey
		Parameter (c)			
80	(1,1)	0.0140	-0.0018	0.0189	0.0208
		(0.1084)	(0.1218)	(0.1280)	(0.1263)
	(1,2)	0.0184	0.0047	0.0195	0.0186
		(0.0945)	(0.1147)	(0.0783)	(0.0789)
	(2,1)	0.0443	0.0072	0.0540	0.0446
		(0.2234)	(0.2583)	(0.2539)	(0.2431)
	(2,2)	0.0373	0.0067	0.0361	0.0343
		(0.1890)	(0.2289)	(0.1576)	(0.1615)
	(2,5)	0.0362	0.0060	0.0259	0.0287
		(0.1719)	(0.2158)	(0.0807)	(0.0759)
	(5,2)	0.0535	-0.0325	0.0863	0.0632
		(0.4326)	(0.5360)	(0.3672)	(0.3848)
		Parameter (k)			
80	(1,1)	0.0101	0.0071	0.0187	0.0178
		(0.1265)	(0.1272)	(0.1511)	(0.1485)
	(1,2)	0.0184	0.0047	0.0195	0.0186)
		(0.0945)	(0.1147)	(0.0783)	(0.0789)
	(2,1)	0.0035	-0.0006	0.0083	0.0073
		(0.1266)	(0.1268)	(0.1521)	(0.1494)
	(2,2)	0.0242	-0.0013	0.0113	0.0121
		(0.2361)	(0.2503)	(0.3001)	(0.2956)
	(2,5)	0.1388	0.0299	0.0037	0.0104
		(0.7470)	(0.9150)	(0.7322)	(0.7214)
	(5,2)	0.0462	0.0186	0.0506	0.0487
		(0.2548)	(0.2696)	(0.3039)	(0.3050)

In Tables IV-VI we give the simulation results for the case with outliers. The outliers are generated by shifting the largest observations to the right in the X direction. For the small sample size $(n = 20)$ we have one outlier, for the moderate sample size $(n = 40)$ we have two outliers and for the large sample size $(n = 80)$ we have four outliers.

Table IV presents the simulation results for the case with one outlier. From this table, we can observe that the robust estimators have lower bias and *RMSE* for the parameter c. Concerning the parameter k, the robust estimators and the *LS* estimators give similar results. Only for the case $(c,k) = (2,5)$ and $(5,2)$ the robust estimators show superiority to the *LS* estimator. The robust estimators outperform the *ML* estimator in all the cases.

In Table V, we present the simulation results for the case with two outliers. For the parameter c, the robust estimators have lower *RMSE* than the *ML* and the *LS* estimators in all simulation conditions. In general, the robust estimators have the lowest *RMSE* for the parameter k. For example, for the case $(c,k) = (2,2)$ the *ML* and *LS* estimators have the larger bias and RMSE values than the robust estimators.

We summarize the simulation results for the case four outliers in Table VI. From this table, we can also observe that the robust estimators have lower bias and *RMSE* values than the *ML* and the *LS* estimators in almost all the simulation configurations.

In summary, unlike the *ML* and *LS* estimators, the robust estimators show similar performance for the case with and without outliers. Therefore, since the robust estimators are not influenced by the outliers, they can be used instead of *ML* or *LS* estimators.

TABLE IV
THE BIAS AND RMSE (PARENTHESES) FOR $n = 20$ WITH ONE OUTLIER

n	(c,k)	Parameter (c)			
		ML	*LS*	Huber	Tukey
20	(1,1)	-0.0902	-0.0348	-0.0103	0.0048
		(0.2199)	(0.2329)	(0.2175)	(0.2128)
	(1,2)	-0.2729	-0.1028	-0.0473	-0.0333
		(0.2992)	(0.2084)	(0.1294)	(0.1226)
	(2,1)	-0.3837	-0.1629	-0.0885	-0.0550
		(0.5486)	(0.4660)	(0.4208)	(0.4108)
	(2,2)	-0.7613	-0.2905	-0.0892	-0.0544
		(0.7888)	(0.4572)	(0.2633)	(0.2593)
	(2,5)	-1.0811	-0.5426	-0.0831	-0.0108
		(1.0851)	(0.5852)	(0.1361)	(0.1218)
	(5,2)	-2.6302	-1.0417	-0.2051	0.0921
		(2.6589)	(1.2914)	(0.6386)	(0.6412)
		Parameter (k)			
20	(1,1)	-0.2366	-0.0718	-0.0210	-0.0153
		(0.2865)	(0.2212)	(0.2840)	(0.2823)
	(1,2)	-0.7719	-0.3405	-0.0150	0.0004
		(0.7928)	(0.4718)	(0.6392)	(0.6372)
	(2,1)	-0.3712	-0.1082	-0.0151	-0.0102
		(0.3948)	(0.2265)	(0.3132)	(0.3110)
	(2,2)	-1.0549	-0.4778	-0.0523	-0.0233
		(1.0622)	(0.5444)	(0.6014)	(0.6179)
	(2,5)	-3.6616	-2.4906	-0.1620	0.1015
		(3.6630)	(2.5095)	(1.5095)	(1.6367)
	(5,2)	-1.3711	-0.6411	-0.0679	0.0305
		(1.3736)	(0.6715)	(0.6012)	(0.6439)

TABLE V
THE BIAS AND RMSE (PARENTHESES) FOR $n = 40$ WITH TWO OUTLIERS

n	(c,k)	Parameter (c)			
		ML	*LS*	Huber	Tukey
40	(1,1)	-0.1272	-0.0523	-0.0364	-0.0280
		(0.1862)	(0.1706)	(0.1555)	(0.1524)
	(1,2)	-0.2836	-0.1014	-0.0469	-0.0374
		(0.2940)	(0.1633)	(0.0964)	(0.0914)
	(2,1)	-0.4287	-0.1591	-0.1043	-0.0640
		(0.5003)	(0.3617)	(0.3268)	(0.3253)
	(2,2)	-0.7938	-0.3004	-0.1155	-0.0915
		(0.8040)	(0.3768)	(0.1937)	(0.1848)
	(2,5)	-1.0920	-0.5145	-0.0787	-0.0144
		(1.0938)	(0.5384)	(0.1062)	(0.0856)
	(5,2)	-2.7026	-1.0492	-0.2811	-0.0156
		(2.7145)	(1.1938)	(0.5093)	(0.4814)
		Parameter (k)			
40	(1,1)	-0.2371	-0.0749	-0.0342	-0.0279
		(0.2634)	(0.1661)	(0.2058)	(0.2046)
	(1,2)	-0.7768	-0.3403	-0.0873	-0.0729
		(0.7862)	(0.4090)	(0.4078)	(0.4039)
	(2,1)	-0.3714	-0.1086	-0.0450	-0.0362
		(0.3826)	(0.1774)	(0.2078)	(0.2052)
	(2,2)	-1.0516	-0.4637	-0.0841	-0.0521
		(1.0550)	(0.5021)	(0.4132)	(0.4221)
	(2,5)	-3.6583	-2.3912	-0.2269	0.0211
		(3.6589)	(2.4056)	(1.0440)	(1.1047)
	(5,2)	-1.3627	-0.6126	-0.0787	0.0201
		(1.3639)	(0.6319)	(0.4271)	(0.4535)

TABLE VI
THE BIAS AND RMSE (PARENTHESES) FOR $n = 80$ WITH FOUR OUTLIERS

n	(c,k)	Parameter (c)			
		ML	*LS*	Huber	Tukey
80	(1,1)	-0.1372	-0.0543	-0.0420	-0.0325
		(0.1677	(0.1331)	(0.1226)	(0.1179)
	(1,2)	-0.2908	-0.1015	-0.0518	-0.0440
		(0.2961	(0.1389)	(0.0797)	(0.0768)
	(2,1)	-0.4515	-0.1680	-0.1190	-0.0962
		(0.4823	(0.2790)	(0.2423)	(0.2315)
	(2,2)	-0.7975	-0.2733	-0.1053	-0.0843
		(0.8030	(0.3220)	(0.1564)	(0.1449)
	(2,5)	-1.0945	-0.4923	-0.0776	-0.0198
		(1.0953	(0.5065)	(0.0930)	(0.0724)
	(5,2)	-2.7133	-0.9966	-0.3046	-0.0473
		(2.7185)	(1.0709)	(0.4326)	(0.3259)
		Parameter (k)			
80	(1,1)	-0.2387	-0.0767	-0.0486	-0.0394
		(0.2534)	(0.1352)	(0.1502)	(0.1494)
	(1,2)	-0.7816	-0.3429	-0.1135	-0.0920
		(0.7864)	(0.3806)	(0.3103)	(0.3051)
	(2,1)	-0.3705	-0.1095	-0.0584	-0.0487
		(0.3760)	(0.1480)	(0.1523)	(0.1487)
	(2,2)	-1.0528	-0.4494	-0.1138	-0.0831
		(1.0546)	(0.4707)	(0.3086)	(0.3088)
	(2,5)	-3.6610	-2.3437	-0.3151	-0.0799
		(3.6614)	(2.3529)	(0.7721)	(0.7792)
	(5,2)	-1.3640	-0.6047	-0.1234	-0.0270
		(1.3646)	(0.6150)	(0.2958)	(0.2945)

V. REAL DATA EXAMPLE

In this section we will analyze the data set given by [20]. This dataset is also considered by [21] to demonstrate the potential of a new family of distributions obtained by compounding the Burr XII and power series distributions. Further, the same dataset is used by [18] to illustrate the modeling and the capability of the *OBR* estimation method. The data set, which is given in Table VII, contains the failure times of 20 mechanical components. The unit for measurement is 1000 h. The boxplot of the dataset given in Fig. 2 displays a potential outlier in the dataset.

TABLE VII
THE FAILURE TIMES OF 20 MECHANICAL COMPONENTS

0.067	0.068	0.076	0.081
0.085	0.085	0.086	0.089
0.098	0.114	0.114	0.115
0.125	0.131	0.149	0.160

We fit a Burr XII distribution to the failure time dataset and estimate the unknown shape parameters c and k using the *ML*, *LS* and the robust estimation methods. Table VIII gives a summary of fitting the Burr XII distribution obtained from the *ML*, *LS* and the robust estimation methods for this dataset. In Fig. 3, we can see that the fitted density obtained from Tukey estimates comparably better than the other fitted densities in terms of modeling the data. The fitted density obtained from Huber estimates is also better than the fitted density obtained from the *ML* and the *LS* estimates.

TABLE VIII
ML, LS AND ROBUST ESTIMATES FOR THE FAILURE TIME DATASET

Method	$\hat{c}$	$\hat{k}$
ML	1.6924	29.9046
LS	2.2980	96.3514
Huber	2.3330	109.9964
Tukey	2.4665	153.2042

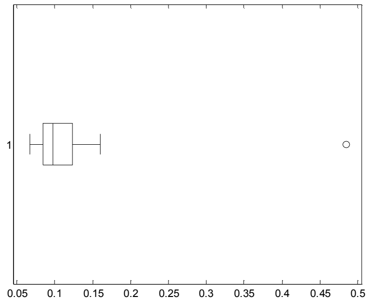

Fig. 2 Boxplot of the failure time dataset

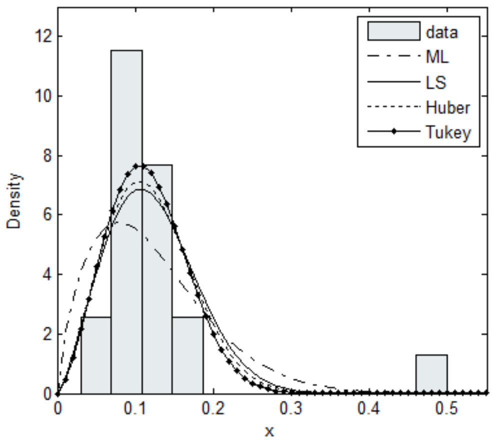

Fig. 3 Histogram with the fitted densities obtained from the *ML*, *LS* and the robust estimation methods

VI. CONCLUSION

In this paper, we have proposed alternative robust estimators for the shape parameters of the Burr XII distribution. We have explored the effect of outliers on the *ML* and the *LS* estimators and the newly proposed robust estimators. The simulation results confirmed that without outliers the robust, the *ML* and *LS* estimators have similar performance. On the other hand, the robust estimators perform better than the *ML* and the *LS* estimators when there are outliers in the data. We have also observed the same results for the real data example. In particular the robust estimator obtained from Tukey's ρ function, is better than the *ML* and the *LS* estimators in terms of coping with the outliers.

Finally, finding the estimators for the Burr XII distribution is a challenging problem. There are many methods introduced to obtain estimators for the shape parameters. Here we can conclude that the robust estimators presented in this paper can be plausible alternative to the estimators given in literature.

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