Discrete Tracking Control of Nonholonomic Mobile Robots: Backstepping Design Approach

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***Abstract*—**In this paper we propose a discrete tracking control of nonholonomic mobile robots with two degrees of freedom. The electromechanical model of a mobile robot moving on a horizontal surface without slipping, with two rear wheels controlled by two independent DC electric, and one front roal wheel is considered. We present backstepping design based on the Euler approximate discrete-time model of a continuous-time plant. Theoretical considerations are verified by numerical simulation.

***Keywords*—**Actuator Dynamics, Backstepping, Discrete-Time Controller, Lyapunov Function, Wheeled Mobile Robot.

I. Introduction

An autonomous wheeled mobile robot is a complex, controllable electromechanical system [1]. The problem of tracking control of wheeled mobile robots has attracted a lot of attention over the past twenty years. The backstepping technique [2]-[5] has been widely used as one of popular strategies for controlling nonholonomic mobile robots considering kinematics and dynamics. Initially there is considered the kinematic model of the system, and by constructing Lyapunov functions there are founded laws of speeds to ensure the stabilization of a given motion. And then constructed the control for the entire system, including both the dynamic model, and using composite Lyapunov function provides the rationale for the observed laws to solve the problem stabilization. The disadvantage of this approach to the synthesis of control is usually a complex nonlinear structure built of law, which leads to the problem of simplifying the structure through the use of new types of Lyapunov functions.

In many practical problems, the control law, realizable by specific actuators cannot be adequately described by means of continuous models, as the availability of digital devices in control systems for mechanical objects, leads to the fact that information between the parts management system is transferred at discrete points in time and in the form of a discrete and executive agencies implement control taking values in some fixed discrete sets. Hence, it is necessary to study of discrete control models.

The purpose of this paper is the constructing a motion stabilizing control of three-wheeled mobile robot with two degrees of freedom on the basis of a sampling system and application of backstepping procedure with Lyapunov function, which allows to simplify the structure of the law found.

The paper is organized as follows: Section II introduces a mathematical model of mobile robot for the problem of tracking control; the proposed control is presented in Section III; in Section IV the conclusion of the paper is presented.

II. Problem Formulation

We consider a mobile robot with two degrees of freedom (Fig. 1). The system consists of a vehicle with two driving wheels mounted on the same axis and a front free wheel. The motion control is achieved by two independent actuators, e.g., DC motors. The kinematics, robot dynamics and actuator dynamics of this robot are described by the following differential equations:

$$\dot{x} = v\cos\theta, \quad \dot{y} = v\sin\theta, \quad \dot{\theta} = \omega \tag{1}$$

$$\begin{aligned} m\dot{v} - m_1 a\omega^2 - \frac{nc}{r}(i_1 + i_2) = 0 \\ J\dot{\omega} + m_1 av\omega - \frac{ncl}{r}(i_1 - i_2) = 0 \end{aligned} \tag{2}$$

$$\begin{aligned} L\frac{di_1}{dt} + Ri_1 + \frac{nc}{r}(v + l\omega) = U_1 \\ L\frac{di_2}{dt} + Ri_2 + \frac{nc}{r}(v - l\omega) = U_2 \end{aligned} \tag{3}$$

where x, y are the coordinates of the point A in the middle of the rear axle of the vehicle, and θ is the angle between the heading direction and the X axis, v is the velocity of the point A, and ω represents the angular velocity of the vehicle, i_1, i_2 are the currents, r is the radius of the back wheels, c is the so-called coefficient of electromechanical interaction, n is the gear ratio of the reduction gearbox, l is the half interval between back wheels, $a = AC$, L is the generalized inductance of the circuit of an electric motor, R is the ohmic resistance of the circuit of a rotor, i_1, i_2 are currents in external circuits of electric motors, U_1, U_2 are the voltages applied to the motors.

$$m = m_1 + 2m_k + 2\frac{J_y}{r^2}$$

$$J = J_1 + 2J_{kz} + (m - m_1)l^2 + m_1 a^2$$

Here m_1 is the mass of the platform, m_k is the total mass of the driving wheel together with the rotor of the electric motor,

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J_1 is the moment of inertia of the chassis with respect to the vertical axis passing through the center of mass C, J_{kz} is the moment of inertia of a driving wheel with respect to the vertical axis, $J_y = J_{ky} + n^2 J_{ry}$ is the "reduced" moment of inertia of a wheel, J_{ky} is the moment of inertia of a driving wheel with respect to the horizontal axis, J_{ry} is the moment of inertia of the rotor of an electric motor.

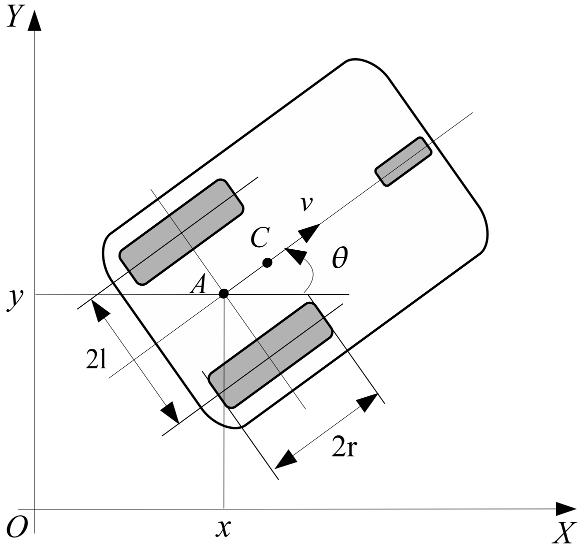

Fig. 1 The mobile robot with two degrees of freedom

The robot is required to follow a trajectory generated by an exosystem, i.e., a fictitious "reference robot":

$$\dot{x}_r = v_r(t)\cos\theta_r, \quad \dot{y}_r = v_r(t)\sin\theta_r, \quad \dot{\theta}_r = \omega_r(t)$$

where the functions $v_r(t)$ and $\omega_r(t)$ are given reference velocities.

III. Problem Solution

We introduce the deviation from the desired trajectory by

$$\begin{bmatrix} x_e \\ y_e \\ \psi_e \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r - x \\ y_r - y \\ \psi_r - \psi \end{bmatrix} \tag{3}$$

Neglecting the inductance L can obtain the following equation of dynamic model of the robot

$$\begin{cases} \dot{x}_e = \omega y_e - v + v_r(t)\cos\psi_e, \\ \dot{y}_e = -\omega x_e + v_r(t)\sin\psi_e, \\ \dot{\psi}_e = \omega_r(t) - \omega, \\ m\dot{v} - m_1 a\omega^2 + \dfrac{2n^2c^2}{r^2R} v = \dfrac{nc}{rR} U_3, \\ J\dot{\omega} + m_1 a v\omega + \dfrac{2n^2c^2l^2}{r^2R}\omega = \dfrac{ncl}{rR} U_4 \end{cases} \tag{4}$$

where $U_3 = U_1 + U_2$ and $U_4 = U_1 - U_2$.

Difference equations in deviations for discrete models based on the Euler approximation take the form

$$\begin{cases} x_e[k+1] = x_e[k] + T(\omega[k]y_e[k] - v[k] + v_r[k]\cos\psi_e[k]), \\ y_e[k+1] = y_e[k] + T(-\omega[k]x_e[k] + v_r[k]\sin\psi_e[k]), \\ \psi_e[k+1] = \psi_e[k] + T(\omega_r[k] - \omega[k]) \\ v[k+1] = v[k] + T(\dfrac{m_1 a}{m}\omega^2[k] - \dfrac{2n^2c^2}{r^2Rm} v[k] + \dfrac{nc}{rRm} U_3[k]), \\ \omega[k+1] = \omega[k] + T(-\dfrac{m_1 a}{J} v[k]\omega[k] - \dfrac{2n^2c^2l^2}{r^2RJ}\omega[k] + \dfrac{ncl}{rRJ} U_4[k]) \end{cases} \tag{5}$$

We define the state vector $(\eta, \xi) \in R^5$ of the system (5) as

$$\eta = (\eta_1, \eta_2, \eta_3)^T, \ \xi = (\xi_1, \xi_2)^T,$$

$$\eta_1 = y_e, \ \eta_2 = x_e - \gamma y_e, \ \eta_3 = \psi_e, \qquad \xi_1 = v, \ \xi_2 = \omega.$$

($\gamma = \gamma[k]$ is the some bounded function),

Then in the new variables, (5) takes the form, consisting of two subsystems

$$\begin{cases} \eta_1[k+1] = (1 - \gamma[k]T\xi_2[k])\eta_1[k] - T\xi_2[k]\eta_2[k] + Tv_r[k]\sin\eta_3[k], \\ \eta_2[k+1] = (\gamma^2[k]+1)T\xi_2[k]\eta_1[k] + (1+\gamma[k]T\xi_2[k])\eta_2[k] - T\xi_1[k] + \\ +Tv_r[k](\cos\eta_3[k] - \gamma[k]\sin\eta_3[k]) + (\gamma[k] - \gamma[k+1])(Tv_r[k]\sin\eta_3[k] + \\ +(1 - \gamma[k]T\xi_2[k])\eta_1[k] - T\xi_2[k]\eta_2[k]), \\ \eta_3[k+1] = \eta_3[k] + T(\omega_r[k] - \xi_2[k]), \\ \xi_1[k+1] = \xi_1[k] + T(\dfrac{m_1 a}{m}\xi_2{}^2[k] - \dfrac{2n^2c^2}{r^2Rm}\xi_1[k] + \dfrac{nc}{rRm} U_3[k]), \\ \xi_2[k+1] = \xi_2[k] + T(-\dfrac{m_1 a}{J}\xi_1[k]\xi_2[k] - \dfrac{2n^2c^2l^2}{r^2RJ}\xi_2[k] + \dfrac{ncl}{rRJ} U_4[k]) \end{cases} \tag{6}$$

The system (6) is represented as a cascade of two subsystems with state vectors η and ξ, respectively. The input signal for the first subsystem of a cascade is the state vector of the second subsystem. To construct stabilizing controls U_3 and U_4 for (6) use a recursive procedure of backstepping method [2], which is applied to this problem means that first we find the law $\xi[k] = \varphi(kT, \eta[k])$ with smooth function $\varphi: R \times R^3 \to R^2$, which stabilizes the zero position $\eta = 0$ of the first subsystem (6), and the corresponding Lyapunov function. Then, by constructing a composite Lyapunov function for the whole system (6), we obtain the desired control. Put $|\eta_3| < \delta_3$ and

$$\begin{aligned}&\varphi_1[k,\eta[k]] = v_r[k] + 2\gamma[k](\omega_r[k] + \mu\eta_3[k])\eta_2[k] + \\ &+(1+\gamma^2[k])(\omega_r[k] + \mu\eta_3[k])\eta_1[k], \\ &\varphi_2[k,\eta[k]] = \omega_r[k] + \mu\eta_3[k]\end{aligned} \quad (7)$$

where the functions $\omega_r[k]$ and $\gamma[k]$ and constant μ satisfy

$$\begin{aligned}&|\omega_r[k]| \le \omega_{r\max},\ 0 < \mu < \frac{2}{T}, \mu \ne \frac{1}{T}, \\ &|\gamma[k] - \gamma[k+1]| < \delta, \\ &0 < \gamma[k]sign(\omega_r[k]) < \frac{2}{T(1+\delta)\omega_{r\max}}\ \ if\ \ \omega_r[k] \ne 0, \\ &0 < \gamma[k]sign(\eta_3[k]) < \frac{2}{T(1+\delta)\mu\delta_3}\ \ if\ \ \omega_r[k] = 0\end{aligned} \quad (8)$$

with control

$$\xi_1[k] = \varphi_1(kT,\eta[k]),\ \xi_2[k] = \varphi_2(kT,\eta[k])$$

kinematic part of the system (6) takes the form

$$\begin{cases}\eta_1[k+1] = (1-\gamma[k]T(\omega_r[k]+\mu\eta_3[k]))\eta_1[k] \\ -T(\omega_r[k]+\mu\eta_3[k])\eta_2[k] + Tv_r[k]\sin\eta_3[k], \\ \eta_2[k+1] = (1-\gamma[k]T(\omega_r[k]+\mu\eta_3[k]))\eta_2[k] \\ +Tv_r[k](\cos\eta_3[k] - \gamma[k]\sin\eta_3[k] - 1) + \\ +(\gamma[k]-\gamma[k+1])(Tv_r[k]\sin\eta_3[k] + \\ (1-\gamma[k]T(\omega_r[k]+\mu\eta_3[k]))\eta_1[k] - \\ -T(\omega_r[k]+\mu\eta_3[k])\eta_2[k]), \\ \qquad \eta_3[k+1] = (1-\mu T)\eta_3[k]\end{cases} \quad (9)$$

To prove the uniform asymptotic stability of the zero solution $\eta_1 = \eta_2 = \eta_3 = 0$ of the system (9) take the Lyapunov function $V = V(\eta_1,\eta_2,\eta_3)$ as

$$\begin{aligned}&V = \max(|\eta_1|, \varepsilon_1|\eta_2|, \varepsilon_2|\eta_3|), \\ &\varepsilon_1, \varepsilon_2 = const > 0\end{aligned}$$

We have the following estimate $V[k+1] \le \| C(k,\eta[k]) \| V[k]$, where the matrix $C(k,\eta[k])$ has the form

$$\begin{pmatrix} c_{11}[k] & c_{12}[k] & c_{13}[k] \\ c_{21}[k] & c_{22}[k] & c_{23}[k] \\ 0 & 0 & c_{33}\end{pmatrix}$$

$$\begin{aligned}&c_{11}[k] = 1-\gamma[k]T(\omega_r[k]+\mu\eta_3[k]), \\ &c_{12}[k] = -\frac{T}{\varepsilon_1}(\omega_r[k]+\mu\eta_3[k]), \\ &c_{13}[k] = \frac{T}{\varepsilon_2}v_r[k]\int_0^1 \cos(s\eta_3[k])ds, \\ &c_{21}[k] = (\gamma[k]-\gamma[k+1])c_{11}[k] \\ &c_{22}[k] = 1-\gamma[k]T(\omega_r[k]+\mu\eta_3[k]) + (\gamma[k]-\gamma[k+1])c_{12}[k], \\ &c_{23}[k] = -\frac{T}{\varepsilon_2}v_r[k]\int_0^1 (\sin(s\eta_3[k]) + \gamma\cos(s\eta_3[k]))ds + \\ &+(\gamma[k]-\gamma[k+1])c_{13}[k], \\ &c_{33} = 1-\mu T,\end{aligned}$$

and

$$\| C(k,\eta[k]) \| = \max_{i=1,2,3}\sum_{j=1}^{3} |c_{ij}[k]|$$

We have the following comparison equation

$$u[k+1] = \| C(k,\eta[k]) \| u[k]$$

Denote

$$\begin{aligned}&\varepsilon_3 = \frac{T}{\varepsilon_2}\max\{\frac{\varepsilon_2}{\varepsilon_1}(\omega_{r\max} + \mu\delta_3) + v_{r\max}\max_{|\eta_3|<\delta_3}|\int_0^1 \cos(s\eta_3)ds|, \\ &v_{r\max}(\max_{|\eta_3|<\delta_3}|\int_0^1 \sin(s\eta_3)ds| + \\ &+\gamma[k]\,|\gamma[k]-\gamma[k+1]|\max_{|\eta_3|<\delta_3}|\int_0^1 \cos(s\eta_3))ds|)\}\end{aligned}$$

Then we obtain the following estimate for the norm of the matrix $C(k,\eta[k])$

$$\begin{aligned}&\| C(k,\eta[k]) \| < \max(|1-\gamma[k]T\omega_r[k]|(1+|\gamma[k]-\gamma[k+1]|) + \\ &+\gamma[k]T\mu|1-\mu T|^k\ \delta_3 + \varepsilon_3, |1-\mu T|), \\ &if\ \ \omega_r[k] \ne 0; \\ &\qquad \| C(k,\eta[k]) \| < \max(|1-\gamma[k]T\mu\eta_3[k]|(1+ \\ &\qquad +|\gamma[k]-\gamma[k+1]|) + \varepsilon_3, |1-\mu T|), \\ &\qquad if\ \ \omega_r[k] = 0;\end{aligned}$$

Choosing constants $\varepsilon_1, \varepsilon_2 = const > 0$ large enough, under the conditions (8) we obtain that there exists a number ε_0, $0 < \varepsilon_0 < 1$, that we have the estimate

$$\begin{aligned}&V[k+1] < (\varepsilon_0 + \gamma[k]T\mu|1-\mu T|^k\ \delta_3)V[k], \\ &\forall k \ge 0\end{aligned}$$

Given the condition of constant μ from (8), find that the comparison equation

$$u[k+1] = (\varepsilon_0 + \gamma[k]T\mu|1-\mu T|^k\ \delta_3)u[k]$$

is uniformly asymptotically stable.

Thus, it was found that the law $\xi[k]=\varphi(kT,\eta[k])$ provides stabilization to the uniform asymptotic stability of the zero position $\eta=0$ of the first subsystem (6).

In [5] by using the method of backstepping for the part of the kinematic system (5) there was built control of the form

$$v[k,\eta[k]]=v_r[k]+\gamma x_e[k]+ \\ +\frac{(\gamma^2+\omega_r^{\ 2}[k]-\varepsilon\gamma)x_e[k]-(2\gamma\omega_r[k]-\varepsilon\omega_r^{\ 3}[k])y_e[k]}{2(1/T-\gamma)+\varepsilon\omega_r^{\ 2}[k]}, \quad (10) \\ \omega[k,\eta[k]]=\omega_r[k]+\mu\psi_e[k]$$

which solves the problem of stabilization to the uniform asymptotic stability of the zero solution $x_e=y_e=\psi_e=0$ of the first subsystem (5).

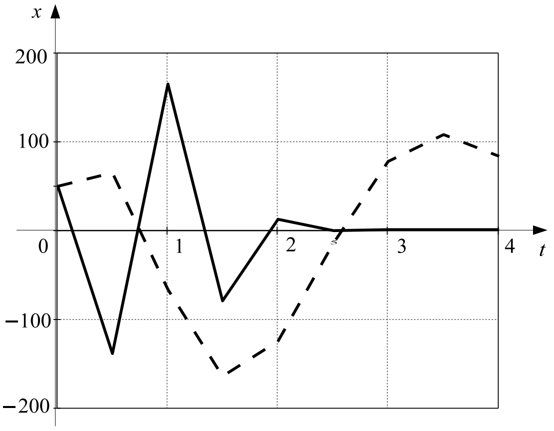

Fig. 2 The simulation results for the controls (7) and (10) and values
$x_0=50,\ T=0.5\,c,\ \mu=1,\gamma=1,$
$\delta_1=50,\ \delta_2=20,\ \delta_3=2.5,\ N=9$

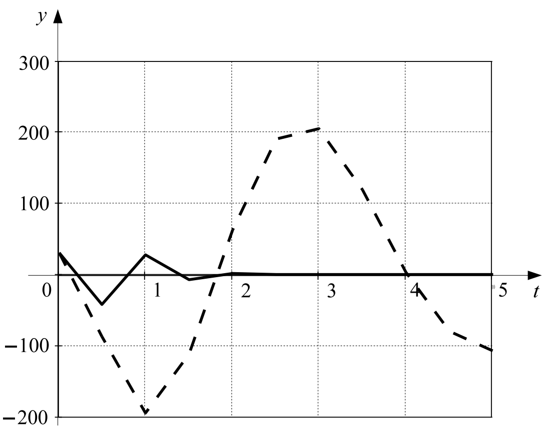

Fig. 3 The simulation results for the controls (7) and (10) and values
$y_0=20,\ T=0.5\,c,\ \mu=1,\gamma=1,$
$\delta_1=50,\ \delta_2=20,\ \delta_3=2.5,\ N=11$

Figs. 2 and 3 show the results of modeling of the kinematic system (6) with controls (7) and (10) for $0\le k\le N$. On Fig. 2 and 3, the solid line shows a process obtained by the control (7) and a dotted – by the control (10). As can be seen, control (7) provides a higher rate of convergence compared with (10).

Using composite Lyapunov function

$$V_c=\max\{V(\eta_1,\eta_2,\eta_3),|\xi_1-\varphi_1[k,\eta]|/\alpha, \\ |\xi_2-\varphi_2[k,\eta]|/\alpha\},\quad \alpha=const>0$$

we can show that control of the form

$$U_3=\frac{rRm}{nc}\Big(\frac{2n^2c^2}{r^2Rm}\xi_1[k]-\frac{m_1a}{m}\xi_2^{\ 2}[k]-a_1(\xi_1[k]-\varphi_1[k,\eta[k]])+ \\ +\frac{\varphi_1[k+1,\eta[k+1]]-\varphi_1[k,\eta[k]]}{T}\Big), \\ U_4=\frac{rRJ}{ncl}\Big(\frac{m_1a}{J}\xi_1[k]\xi_2[k]+\frac{2n^2c^2l^2}{r^2RJ}\xi_2[k]- \\ -a_1(\xi_2[k]-\varphi_2[k,\eta[k]])+\frac{\varphi_2[k+1,\eta[k+1]]-\varphi_2[k,\eta[k]]}{T}\Big), \\ (0<a_1<\frac{2}{T}) \quad (11)$$

provides a uniform asymptotic stability of the zero solution of system (6). Indeed, we make the change of variables

$$z_1=\xi_1-\varphi_1[k,\eta],\quad z_2=\xi_2-\varphi_2[k,\eta]$$

The system (6) with control (11) can be written as

$$\begin{cases}\eta_1[k+1]=(1-\gamma T\varphi_2[k])\eta_1[k]-T\varphi_2[k]\eta_2[k]+ \\ +Tv_r[k]\sin\eta_3[k]-\gamma Tz_2[k])\eta_1[k]-Tz_2[k]\eta_2[k], \\ \eta_2[k+1]=(\gamma^2+1)T\varphi_2[k]\eta_1[k]+ \\ +(1+\gamma T\varphi_2[k])\eta_2[k]-T\varphi_1[k]+ \\ +Tv_r[k](\cos\eta_3[k]-\gamma\sin\eta_3[k])+ \\ +(\gamma^2+1)Tz_2[k]\eta_1[k]+\gamma Tz_2[k])\eta_2[k]-Tz_1[k], \\ \eta_3[k+1]=(1-\mu T)\eta_3[k]-Tz_2[k], \\ z_1[k+1]=(1-a_1T)z_1[k],\quad z_2[k+1]=(1-a_1T)z_2[k]\end{cases}$$

For composite Lyapunov function along the solution of the system with the initial condition $V_c[0]<\delta=const>0$ we obtain

$$V_c[k+1]<\max\{\varepsilon_0+\gamma T\mu\,|1-\mu T|^k\ \delta/\varepsilon_2+ \\ +\alpha T\max[\delta(\gamma+1/\varepsilon_1),\delta(\gamma^2+\gamma/\varepsilon_1+1)+1], \\ |1-a_1T|\}V_c[k],\ \forall k\ge 0$$

Then, provided that

$$\varepsilon_0+\alpha T\max[\delta(\gamma+1/\varepsilon_1),\delta(\gamma^2+\gamma/\varepsilon_1+1)+1]<1$$

we find that the comparison equation

$$u[k+1] = \max\{\varepsilon_0 + \gamma T \mu |1-\mu T|^k \delta + \\ +\alpha T \max[\delta(\gamma + 1/\varepsilon_1), \delta(\gamma^2 + \gamma/\varepsilon_1 + 1) + 1], \\ |1 - a_1 T|\} u[k]$$

is uniformly asymptotically stable. Hence we obtain the uniform asymptotic stability of the zero solution of (6).

IV. Conclusion

In the paper we proposed a stabilizing motion control law of three-wheeled mobile robot with two degrees of freedom which is justified the technique of the solving the problem of stabilization of nonlinear time-varying systems with piecewise constant control on the basis of a sampling system and application of backstepping method. This method is based on the representation of the entire system as a cascade connection of subsystems and on the following synthesis of nonlinear control law on the basis of constructing of Lyapunov functions for each subsystem. Effectiveness of the construction of the control law is that, firstly, it is applicable for a wide class of program motions, secondly, is easily implemented in software, thirdly, it allows to determine for each case the most appropriate set of control parameters. The novelty of this technique is the use of Lyapunov functions of the form of vector norms and difference comparison equations, effective in constructing of laws of discrete controls for considered systems with higher speed of convergence, the expansion of the domain of attraction of solutions, simplifying the control structure in comparison with known results. We consider electromechanical model of a mobile robot moving on a horizontal surface without slipping, with two rear wheels are controlled by two independent DC electric, and front roal wheel. On the problem of synthesis of a piecewise constant control of a continuous system there is used an approach based on sampling systems with use of Euler approximation and construction of stabilizing discrete control laws for discrete model. On the basis of the recurrent procedure of backstepping method consisting in the transition from the control synthesis problem for the kinematic model to the problem of constructing controlling signals for dynamic model of the robot found a piecewise constant control law that practically stabilizes the set of unsteady motion of the robot. The results of numerical simulations confirming the effectiveness of the proposed control law are presented.

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