Efficient Frontier - Comparing Different Volatility Estimators

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***Abstract*—**Modern Portfolio Theory (MPT) according to Markowitz states that investors form mean-variance efficient portfolios which maximizes their utility. Markowitz proposed the standard deviation as a simple measure for portfolio risk and the lower semi-variance as the only risk measure of interest to rational investors. This paper uses a third volatility estimator based on intraday data and compares three efficient frontiers on the Croatian Stock Market. The results show that range-based volatility estimator outperforms both mean-variance and lower semi-variance model.

***Keywords*—**Variance, lower semi-variance, range-based volatility.

I. INTRODUCTION

MODERN Portfolio Theory was introduced by Nobel Laureate Harry Markowitz [15] in his seminal paper which changed the way portfolios were managed until then. This theory focuses on portfolio diversification and risk control. Investors form portfolios according to the mean variance efficiency criteria. This means that investors maximize their return across all possible portfolios and accept the risk according to their risk aversion. Markowitz describes a risk averse investor as a subject who prefers a higher return versus a lower return and who at the same time is prepared to accept more risk if such investment increases the expected return. Such an investor optimizes the expected portfolio return given the portfolio risk. Modern portfolio theory is based on the efficient frontier (EF) of investments, i.e. the spine of portfolios with maximum expected return across all possible portfolios given a certain amount of portfolio risk. The portfolio risk is of crucial information to the investor and therefore needs to be quantified. The volatility of the portfolio return is often considered as the risk of concern. Since volatility is not observable it needs to be estimated. Markowitz proposed to quantify portfolio risk by means of the volatility of financial assets. He used the standard deviation of financial assets as a simple measure of risk and the lower semi-variance as the more complex estimator. The lower semi-variance is according to Markowitz the only volatility estimator in which a rational investor might be interested in. On the other side the standard deviation has become one of the most popular risk estimators in practice due to the simplicity in using and understanding this measure in portfolio management. However, both estimators use only one single daily observable price change to determine the volatility of financial returns. An alternative volatility estimator used in this paper is based on high frequency data. Volatility estimated by means of high frequency data is also called realized volatility and can be considered unbiased. In practice, however, the implementation of high frequency data is limited by several reasons. First, high frequency data is not available for all securities. This is especially true for securities traded in emerging markets where the trading volume is often insufficient as the data frequency becomes smaller. Secondly, as the frequency becomes smaller microstructure effects emerge which induce an upward bias in the estimated volatility. Thirdly, there is a serious calculation complexity due to the extensive amount of data that is required for estimating the daily volatility or the variance-covariance matrix. For example to calculate the volatility of 250 trading days based on 5-minute interval observations around 24.000 intraday price observations are required. Moreover, for estimating the EF of a portfolio consisting of 20 assets more than a million observations will be required. A more practical methodology to estimate the intraday volatility is by means of open, high, low and closing prices (OHLC). This paper uses the Parkinson [17] range-based volatility estimator for extreme price jumps, which are characteristic for emerging markets like Croatia. According to empirical research performed by [4] the range-based volatility estimator is the least biased volatility estimator using OHLC data when measuring the volatility of the Croatian stock market. A significant shortcoming however, of the range-based volatility estimator is that no multivariate analogue of the intraday range exists, which means that the estimation of the variance-covariance matrix is not straightforward. A simple estimator of the conditional variance-covariance matrix of returns was proposed by [12]. This methodology is used to construct the EF based on the range-based volatility estimator. This paper compares EF based on 3 different volatility estimators using a portfolio of stocks from the Croatian Stock Market.

The outline of the remainder of the paper is as follows. Section II reviews the literature on modern portfolio theory with focus on the literature on different approaches to estimating the volatility. Section III describes the modern portfolio theory, which is the basis of this research. Section IV presents the lower semi-variance approach in estimating the efficient frontier and Section V the intraday volatility approach. The stock price data is described in Section VI. The results of the empirical research are presented in Section VII. Section VIII concludes.

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II. Review of Previous Research

In his seminal paper, [15] describes Modern Portfolio Theory in a quantitative model that solves the complex problem of capital allocation across assets in such a way that it minimizes the variance of the portfolio given an expected return. Markowitz proposed a simple square root optimisation that results in the mean-variance efficient portfolio and suggested to use the standard deviation for estimating the portfolio volatility. The standard deviation is a statistically correct estimator of the volatility of returns if the observed time series are derived from a normal distribution. This, however, is not always the case. One of the stylized facts of financial returns as described in [7] is the non-normality of financial returns. The distribution often 'suffers' from positive skewness and leptokurtosis. Other stylized facts include amongst others heteroscedasticity and time varying correlations of financial returns. Therefore the standard deviation, which assumes normality by default, is expected to underestimate the true volatility of the distribution. Motivated by the definition of risk, as a financial loss or downside risk, [16] proposes a new definition of risk considering only the negative results.

The lower semi-variance measures the dispersion of the returns below a given target return. Markowitz explains that the usage of the lower semi-variance is justified by two reasons. Firstly, rational investors are only interested in limiting the volatility that can cause a negative result. Secondly, if the financial time series are not normally distributed then the standard deviation will underestimate the true risk of the portfolio. In these cases the lower semi-variance, as a measure of downside risk, should be used instead. It is shown in [14] that there is a great support in the market for using the lower semi-variance as a risk measure. Investors are more sensitive to losses below a certain threshold then to gains beyond a certain threshold. In [5] the formula for lower semi-variance is generalized and defined as the lower partial moment (LPM). Four different LPM volatility estimators are compared in [13] and it is shown that the LPM proposed by Markowitz is suitable for controlling risk when the distribution of the assets is not normal.

Both estimators that were proposed by Markowitz use only one single daily price observation to determine the variance-covariance matrix. This means that all other price observations that are available when high frequency data a used are ignored.

One of the recent theories focusing on volatility estimators are described in the literature of high frequency data. The realized volatility estimator is proposed in [8], which is the squared sum of intraday returns. According to [3] this volatility estimator is theoretically unbiased when the frequency sample goes to zero, but will in turn induce microstructure effects. The realized volatility is estimated by using all market available intraday information. Another practical disadvantage of this method is that it requires an extensive amount of intraday price observations for estimating an EF. A reasonable alternative to using high frequency data is to use volatility estimators that require only 4 standard available intraday price observations, i.e. the OHLC estimators. OHLC estimators, generally, assume that asset prices follow a Geometric Brownian Motion (GBM) i.e. the price of the asset on day *t* is independent of the price of the same asset on day *t*-1 and that the price of the assets are stochastic through time. GBM without drift is assumed in [17] and it proposes a range based volatility estimator. This estimator uses the maximum difference between the maximum and the minimum intraday price for estimating the volatility. The open and closing prices are included in [11] and they propose an estimator, which uses all four OHLC intraday price observations. An estimator that follows a GBM with drift is proposed in [18]. This estimator is useful when the drift is non-zero. Significant differences between OHLC estimators that are popular in the literature are found in [9] and they conclude that the choice of the OHLC volatility estimator is important. OHLC volatility estimators that are popular in the literature are compared in [4], against the unbiased high frequency based volatility estimator. They show that the Parkinson range-based volatility estimator is the least biased estimator for estimating the volatility of the Croatian Stock market compared to other OHLC volatility estimators. The comparison is performed against the high frequency based realized volatility which is the theoretically unbiased volatility estimator. The data used in their research spans a period of 5 years and includes the recent credit and bank crisis of 2007 and 2008. They confirm the findings of [9] by means of loss functions and time varying conditional correlations and conclude that the OHLC volatility estimators are significantly different from each other. This paper follows the results of [4] and uses the Parkinson range-based volatility estimator in estimating the intraday volatility. The conditional variance-covariance matrix proposed in [12] is used to construct the EF by means of mean-variance.

EF are compared in [10], [19] and [20] based on the standard deviation and the lower semi-variance and conclude that it is possible to construct an EF based on the lower semi-variance that lies on the left side of the EF based on the simple standard deviation. They conclude that this EF is stochastically dominant compared to the standard deviation proposed by Markowitz. It is possible to reduce the risk of a portfolio by using the lower semi-variance as a measure of the portfolio volatility [10]. According to [6], it is not possible to compare EF based on different volatility estimators since the risk estimators are not identical, i.e. the x-axis on the mean-variance coordinate system is different. They conclude that the only meaningful way of comparing EF is by ex-post analysis and that the location of the EF on the mean-variance coordinate system does not add valuable information.

This research compares EF based on different volatility estimators: standard deviation, lower semi-variance and the intraday volatility estimator. The questions of interest are whether the volatility estimator influences the location of the efficient frontier on the mean-variance coordinate system and whether the location of the efficient frontier on the mean-variance coordinate system determines the performance of the efficient portfolios by the ex-post analysis.

III. Markowitz Modern Portfolio Theory

According to Modern Portfolio Theory (MPT) investors use mean-variance optimization to construct an efficient portfolio. MPT relies on the following assumptions: the investment horizon is one period (one month, one year, etc.); investors optimize their expected return across all possible portfolios; the expected portfolio return depends on the expected return and the risk of the investment; investors are rational and prefer a higher return compared to a lower return, and also have aversion to risk; there is no tax, no inflation and there is no transaction or other costs involved; all investors have free and unlimited access to relevant information at the same time; all stocks are infinitely divisible. This set of assumptions creates a theoretical world in which investors operate according to MPT. This world is different from the real world, but incorporates almost all elements average investors take into account when making investment decisions.

According to MPT investors will spread their portfolio to divers or control the risk and at the same time they want to maximize their expected return. Optimization is based on mean-variance efficiency, which means that, given a predetermined portfolio risk, investors will choose the portfolio that maximizes their return.

The standard deviation is a popular volatility estimator that requires only one price observation per day. This estimator is symmetrical and assumes that the returns follow a normal or multivariate normal distribution.

Considering that every efficient portfolio has the highest revenue along with defined rate of risk *c*, mathematically we may define efficient portfolio as follows [1]:

$$\max E(R_\pi) \tag{1}$$

Subject to:

$$\sigma_\pi \le c \tag{2}$$

$$\sum_{i=1}^{n} \pi_i = 1, \pi_i \ge 0, i \in \{1,2,...,n\} \tag{3}$$

The expected portfolio return is defined as

$$E(R_\pi) = \sum_{i=1}^{n} \pi_i E(R_i) = \pi' \cdot E(R) = E(R)' \cdot \pi \tag{4}$$

and the portfolio risk as

$$\sigma_\pi = \sqrt{\pi' \cdot S \cdot \pi} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i \pi_j \sigma_{ij}}\ . \tag{5}$$

$E(R)$ denotes the vector column of expected returns, $E(R_i)$ the expected return of the stock *i*, π the vector column of weights of a stock in a portfolio, π_i the weight of a stock *i* in portfolio π, S the variance-covariance matrix, σ_{ij} the covariance of returns of stocks *i* and *j*, and *n* the number of stocks.

IV. Lower Semi-Variance Approach

When the returns do not follow a normal distribution, the standard deviation often underestimates the true volatility [7]. The lower semi-variance measures the variance below a certain threshold. Only returns that are underperforming this threshold are taken into account in determining the portfolio risk. Portfolios optimized with this volatility estimator are expected to assign less weight to stocks that are underperforming and more weight to stocks whose returns exceed the threshold.

When the risk is measured by lower semi-variance, the portfolio optimization problem becomes the problem of quadratic programming of the following form given in [2]:

$$\min \left(\sum_{t=1}^{T} p_t z_t^2 \right) \tag{6}$$

with constraints:

$$\begin{gathered} z_t \ge -\sum_{j=1}^{n}\left(r_{j,t} - E\left(R_j\right)\right)\pi_j,\ t \in \{1,2,...,T\} \\ z_t \ge 0,\ t \in \{1,2,...,T\} \\ \sum_{j=1}^{n} \pi_j = 1 \\ \sum_{j=1}^{n} \pi_j\, E(R_j) \ge \rho \\ \pi_j \ge 0,\ j \in \{1,2,...,n\}. \end{gathered} \tag{7}$$

$r_{j,t}$ is the return of the share *j* in period *t*, *T* is the number of trading days. $p_t,\ t \in \{1,2,...,T\}$, is probability that the vector of returns of portfolio *π*, $R = (R_1, R_2, ..., R_n)$ takes value $r_t = (r_{1,t}, r_{2,t}, ..., r_{n,t})$. Usually, $p_t = \frac{1}{T}$. Model's variables are $\pi_j,\ j \in \{1,2,...,n\}$ and $z_t,\ t \in \{1,2,...,T\}$.

V. Intraday Volatility Approach

The standard deviation and the lower semi-variance use one single daily price observation in determining the volatility of the portfolio. All other relevant information available to the investor is ignored. Intraday volatility estimators use more daily price observations in computing the volatility. These estimators do not rely on the normal distribution. It is shown in [8] that the unbiased volatility estimator could be constructed by means of high frequency data. The trading volume in emerging markets is often insufficient to ensure high frequency data for all required stocks. Due to limitations, we follow [4] which showed that the Parkinson range-based volatility estimator is the least biased OHLC estimator when estimating the volatility of the Croatian Stock Market. The Parkinson range based volatility estimator uses two intraday price observations to determine the spread: the highest and the lowest intraday observations.

The range-based volatility estimator is given by:

$$\sigma_{ii,t}^{Range} = \frac{1}{4\ln 2} \cdot \ln\left(\frac{H_i}{L_i}\right)^2 \tag{8}$$

In (8) H denotes the highest and L denotes the lowest observed intraday price. Using the highest and the lowest price observations Parkinson proposes a volatility estimator for high volatile markets. This estimator follows a GBM without drift and uses only extreme price movements to calculate the volatility.

The portfolio risk as defined in (5) requires the variance-covariance matrix for input. The off-diagonal elements of the variance-covariance matrix of the range-based volatility estimator are not directly observable. A simple model, that is based on the exponentially weighted moving average (EWMA), developed to estimate the off-diagonal elements of the variance-covariance matrix when using the range-based volatility estimator, is proposed in [12]. This model combines the range-based and the return-based approaches. The return-based volatility estimator is given by:

$$\sigma_{ii,t}^{R} = \ln^2\left(\frac{p_{i,t}}{p_{i,t-1}}\right) \tag{9}$$

where $p_{i,t}$ is the price of stock i on day t.

The estimator is based on the multivariate EWMA model of the conditional variance-covariance matrix given by:

$$\hat{\sigma}_{ij,t}^{R} = \lambda\hat{\sigma}_{ij,t-1}^{R} + (1-\lambda)\sigma_{ij,t-1}^{R}; i, j = 1,...,n \ . \tag{10}$$

where λ is the single decay factor, which is typically set to 0.94, estimated by JP Morgan as the average value of decay factor that minimizes the mean square error of daily out-of-sample conditional volatility forecasts for a wide range of assets. Covariance is given by:

$$\sigma_{ij,t}^{R} = \ln(r_{i,t})\cdot\ln(r_{j,t}) \ . \tag{11}$$

The diagonal and off-diagonal elements of the range-based estimator of the conditional variance-covariance matrix are given by

$$\hat{\sigma}_{ij,t}^{Range} = \begin{cases} \lambda\hat{\sigma}_{ii,t-1}^{Range} + (1-\lambda)\sigma_{ii,t-1}^{Range}; i = j; i, j = 1,...,n \\ \rho_{ij}^{R}\sqrt{\hat{\sigma}_{ii,t}^{Range}\hat{\sigma}_{jj,t}^{Range}}; i \neq j; i, j = 1,...,n \end{cases} \tag{12}$$

where

$$\rho_{ij}^{R} = \frac{\sigma_{ij,t}^{R}}{\sqrt{\hat{\sigma}_{ii,t}^{R}\hat{\sigma}_{jj,t}^{R}}}; i, j = 1,...,n \tag{13}$$

Finally, the elements of the variance-covariance matrix of the range-based volatility model are calculated by

$$\sigma_{ij} = \hat{\sigma}_{ii,t}^{Range}\hat{\sigma}_{jj,t}^{Range}\rho_{ij}^{R} \ . \tag{14}$$

Now, when the range-based variance-covariance matrix is known, we proceed with steps (1) to (5) to calculate the mean variance portfolio.

VI. Data and Descriptive Statistics

The portfolios constructed in this research consist of an investment in 10 stocks from the CROBEX index. The data spans from 12th March 2013 to 13th December 2013 and counts 191 price observations. The following stocks are included: AD Plastik d.d. (ADPL), Atlantska Plovidba d.d. (ATPL), Belje d.d. (BLJE), Djuro Djakovic Holding d.d. (DDJH), Dalekovod d.d. (DLKV), Valamar Adria Holding d.d. (DOMF), Ericsson Nikola Tesla d.d. (ERNT), Hrvatski Telekom d.d. (HT), Ingra d.d. (INGR) and Vupik d.d. (VPIK).

Table I shows the descriptive statistics including the sample size, minimum, maximum, expected returns, the volatility at the end of the period for each volatility estimator, skeweness, kurtosis and the Jarque-Berra test for normality of returns. The descriptive statistics shows that all assets show asymmetric behaviour and leptokurtosis, i.e. deviation from the normal distribution. The Jarque-Berra test shows that none of the stocks follows a normal distribution.

Fig. 1 shows values of the three observed volatility estimators for each stock. It can be concluded that, on average, the highest volatility is estimated by standard deviation and that the lowest volatility is estimated by the lower semi-standard deviation. However, the range-based volatility estimator yields mixed results, i.e. for certain stocks it estimates volatility higher than the standard deviation does, and sometimes it provides volatility of a stock lower than the lower semi-standard deviation does.

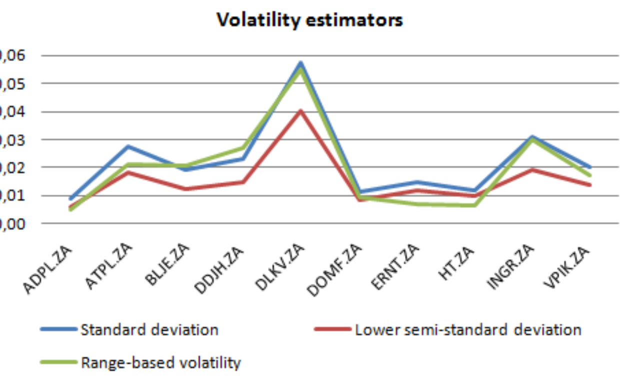

Fig. 1 Volatility estimators

VII. Empirical Results

In this paper the location of the EF on the mean-variance coordinate system and the performance of three different volatility estimators in the ex-post or out-of sample analysis are compared.

In the first part of the analysis, the EF is computed at 20 different risk levels for the portfolios using mean-variance, lower semi-variance and intraday range-based volatility approach. The appropriate weights, returns and standard deviations are presented in Tables II-IV.

TABLE I
DESCRIPTIVE STATISTICS

	ADPL	ATPL	BLJE	DDJH	DLKV	DOMF	ERNT	HT	INGR	VPIK
N	190	190	190	190	190	190	190	190	190	190
Min	-0,0284	-0,0648	-0,0520	-0,0573	-0,3075	-0,0341	-0,1278	-0,1133	-0,0750	-0,0735
Max	0,0412	0,0731	0,1010	0,0825	0,2597	0,0289	0,0422	0,0329	0,1443	0,0724
Expected return	0,00000	0,00133	-0,00203	-0,00145	-0,00233	-0,00031	-0,00025	-0,00111	-0,00154	-0,00152
Variance	0,00007	0,00077	0,00037	0,00053	0,00328	0,00013	0,00021	0,00014	0,00094	0,00040
Standard deviation	0,00864	0,02768	0,01920	0,02292	0,05729	0,01140	0,01452	0,01177	0,03071	0,02002
Lower semi-variance	0,00004	0,00033	0,00016	0,00022	0,00162	0,00007	0,00014	0,00010	0,00036	0,00019
Lower semi-standard deviation	0,00604	0,01810	0,01251	0,01481	0,04025	0,00823	0,01183	0,00998	0,01909	0,01387
Range-based volatility	0,00477	0,02111	0,02043	0,02716	0,05491	0,00947	0,00675	0,00619	0,03015	0,01737
Skeweness	0,24	0,49	1,04	0,69	-0,18	-0,19	-3,44	-4,51	1,21	0,10
Kurtosis	3,66	0,16	5,04	1,37	7,20	0,39	31,17	43,18	4,37	1,65
Jarque-Berra	5,18	71,51	66,93	36,02	140,47	55,01	6659,13	13422,71	61,05	14,83

TABLE II
EFFICIENT PORTFOLIOS USING MEAN-VARIANCE MODEL

ADPL	ATPL	BLJE	DDJH	DLKV	DOMF	ERNT	HT	INGR	VPIK	St. dev. (%)	Return (%)
0,3453	0,0590	0,0347	0,0402	0,0020	0,1867	0,1208	0,1765	0,0000	0,0348	0,5680	-0,0392
0,5818	0,1680	0,0000	0,0000	0,0000	0,1363	0,1139	0,0000	0,0000	0,0000	0,7000	0,0153
0,6344	0,2279	0,0000	0,0000	0,0000	0,0516	0,0862	0,0000	0,0000	0,0000	0,8000	0,0266
0,6615	0,2772	0,0000	0,0000	0,0000	0,0000	0,0613	0,0000	0,0000	0,0000	0,9000	0,0354
0,6547	0,3257	0,0000	0,0000	0,0000	0,0000	0,0196	0,0000	0,0000	0,0000	1,0000	0,0429
0,6277	0,3723	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,1000	0,0496
0,5819	0,4181	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,2000	0,0557
0,5389	0,4611	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,3000	0,0614
0,4979	0,5021	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,4000	0,0669
0,4583	0,5417	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,5000	0,0722
0,4196	0,5804	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,6000	0,0774
0,3817	0,6183	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,7000	0,0824
0,3445	0,6555	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,8000	0,0874
0,3077	0,6923	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,9000	0,0923
0,2713	0,7287	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	2,0000	0,0971
0,2353	0,7647	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	2,1000	0,1019
0,1995	0,8005	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	2,2000	0,1067
0,1640	0,8360	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	2,3000	0,1114
0,0935	0,9065	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	2,5000	0,1208
0,0000	1,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	2,7682	0,1333

TABLE III
EFFICIENT PORTFOLIOS USING LOWER SEMI-VARIANCE MODEL

ADPL	ATPL	BLJE	DDJH	DLKV	DOMF	ERNT	HT	INGR	VPIK	St. dev. (%)	Return (%)
0,5268	0,1216	0,0000	0,0030	0,0000	0,2010	0,0802	0,0675	0,0000	0,0000	0,4500	0,0000
0,5271	0,1218	0,0000	0,0028	0,0000	0,2010	0,0802	0,0672	0,0000	0,0000	0,4502	0,0001
0,5307	0,1238	0,0000	0,0007	0,0000	0,2007	0,0797	0,0643	0,0000	0,0000	0,4518	0,0010
0,5711	0,1484	0,0000	0,0000	0,0000	0,1900	0,0724	0,0181	0,0000	0,0000	0,4724	0,0100
0,6018	0,1950	0,0000	0,0000	0,0000	0,1423	0,0610	0,0000	0,0000	0,0000	0,5074	0,0200
0,6273	0,2513	0,0000	0,0000	0,0000	0,0716	0,0499	0,0000	0,0000	0,0000	0,5636	0,0300
0,6537	0,3074	0,0000	0,0000	0,0000	0,0000	0,0389	0,0000	0,0000	0,0000	0,6374	0,0400
0,6362	0,3417	0,0000	0,0000	0,0000	0,0000	0,0221	0,0000	0,0000	0,0000	0,6805	0,0450
0,6181	0,3762	0,0000	0,0000	0,0000	0,0000	0,0057	0,0000	0,0000	0,0000	0,7282	0,0500
0,5498	0,4502	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,8346	0,0600
0,4748	0,5252	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,9530	0,0700
0,3998	0,6002	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,0788	0,0800
0,3247	0,6753	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,2096	0,0900
0,2872	0,7128	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,2764	0,0950
0,2497	0,7503	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,3441	0,1000
0,1747	0,8253	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,4816	0,1100
0,0996	0,9004	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,6214	0,1200
0,0246	0,9754	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,7629	0,1300
0,0021	0,9979	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,8056	0,1330
0,0000	1,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,8085	0,1333

TABLE IV
EFFICIENT PORTFOLIOS USING RANGE-BASED VOLATILITY MODEL

ADPL	ATPL	BLJE	DDJH	DLKV	DOMF	ERNT	HT	INGR	VPIK	St. Dev. (%)	Return (%)
0,1772	0,0618	0,0443	0,0376	0,0242	0,1955	0,1552	0,2002	0,0410	0,0631	0,3085	-0,0599
0,1990	0,0727	0,0302	0,0218	0,0169	0,1979	0,1724	0,2013	0,0357	0,0522	0,3102	-0,0497
0,2208	0,0836	0,0162	0,0060	0,0095	0,2002	0,1896	0,2025	0,0304	0,0412	0,3152	-0,0396
0,2448	0,0936	0,0023	0,0000	0,0028	0,2023	0,2078	0,1992	0,0217	0,0255	0,3238	-0,0294
0,2742	0,1079	0,0000	0,0000	0,0000	0,2028	0,2233	0,1807	0,0099	0,0012	0,3380	-0,0192
0,3002	0,1341	0,0000	0,0000	0,0000	0,1957	0,2349	0,1344	0,0007	0,0000	0,3611	-0,0091
0,3237	0,1639	0,0000	0,0000	0,0000	0,1882	0,2451	0,0791	0,0000	0,0000	0,3942	0,0011
0,3470	0,1939	0,0000	0,0000	0,0000	0,1807	0,2553	0,0232	0,0000	0,0000	0,4355	0,0113
0,3426	0,2457	0,0000	0,0000	0,0000	0,1647	0,2471	0,0000	0,0000	0,0000	0,4849	0,0214
0,3185	0,3129	0,0000	0,0000	0,0000	0,1428	0,2259	0,0000	0,0000	0,0000	0,5482	0,0316
0,2944	0,3800	0,0000	0,0000	0,0000	0,1209	0,2047	0,0000	0,0000	0,0000	0,6219	0,0418
0,2703	0,4472	0,0000	0,0000	0,0000	0,0989	0,1836	0,0000	0,0000	0,0000	0,7027	0,0519
0,2462	0,5144	0,0000	0,0000	0,0000	0,0770	0,1624	0,0000	0,0000	0,0000	0,7885	0,0621
0,2221	0,5816	0,0000	0,0000	0,0000	0,0551	0,1413	0,0000	0,0000	0,0000	0,8779	0,0723
0,1980	0,6487	0,0000	0,0000	0,0000	0,0332	0,1201	0,0000	0,0000	0,0000	0,9697	0,0824
0,1739	0,7159	0,0000	0,0000	0,0000	0,0113	0,0989	0,0000	0,0000	0,0000	1,0634	0,0926
0,1438	0,7845	0,0000	0,0000	0,0000	0,0000	0,0717	0,0000	0,0000	0,0000	1,1587	0,1028
0,1074	0,8545	0,0000	0,0000	0,0000	0,0000	0,0381	0,0000	0,0000	0,0000	1,2554	0,1129
0,0710	0,9245	0,0000	0,0000	0,0000	0,0000	0,0044	0,0000	0,0000	0,0000	1,3535	0,1231
0,0000	1,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	1,4530	0,1333

The results of the three different volatility estimators show that using different volatility estimator yields with different weights of stocks in a portfolio, which results in different ranges of return and risk. However, the highest risk is found with mean-variance model. It ranges from 0,57%, when diversifying portfolio and investing in all but one stock (INGR) yielding a return of -0,04%, to 2,77% when investing in only one share (ATPL), yielding a return of 0,13%. When considering lower semi-variance, the risk ranges from 0,45%, when diversifying portfolio and investing in ADPL, ATPL, DDJH, DOMF, ERNT and HT yielding 0,00% return, to 1,81% when investing in only one stock (ATPL) yielding return of 0,13%. The lowest risk is measured with range-based estimator ranging from 0,31%, when diversifying risk and investing in all the stocks with different weights yielding the negative return of -0,06%, to 1,45% when investing in ATPL yielding the return of 0,13%. The highest return for all estimators is 0,13% since they all have the same stock in the last portfolio.

It can be concluded that perhaps intraday range-based volatility estimator underestimates the risk compared to the two other volatility estimators. However, the results of the ex-post analysis test the performances of the models.

The computed efficient frontiers are plotted on the mean-variance coordinate system for the mean-variance model, lower semi-variance and intraday range-based volatility model and are presented in Figs. 2-4.

In the second step, an ex-post analysis is performed by investing in a portfolio of stocks using the calculated portfolio weights. The stock returns on the next trading day and the calculated weights are used to calculate for each model the portfolio returns. The results are presented in Table V.

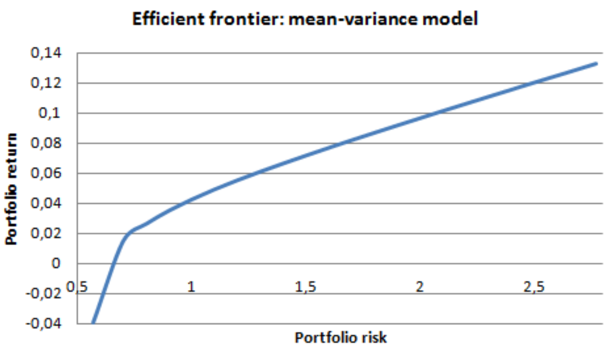

Fig. 2 The efficient frontier based on the mean-variance model

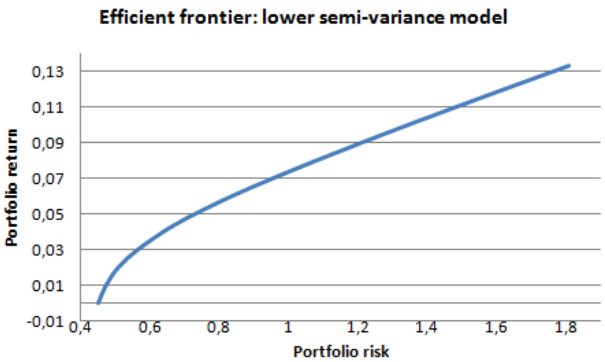

Fig. 3 The efficient frontier based on the lower semi-variance model

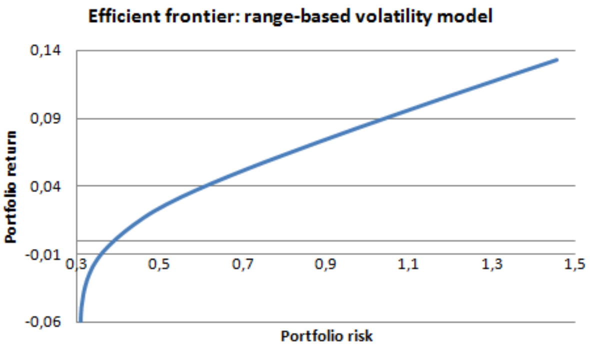

Fig. 4 The efficient frontier based on the range-based estimator model

TABLE V
THE RETURNS (IN %) FOR THREE OBSERVED ESTIMATORS ON 16TH DECEMBER 2013

	mean-variance	lower semi-variance	range-based volatility model
1	0,5039	0,3518	0,8597
2	0,0791	0,3513	0,7389
3	-0,2381	0,3466	0,6181
4	-0,4154	0,3030	0,5066
5	-0,3587	0,1332	0,4460
6	-0,3007	-0,1211	0,3999
7	-0,2405	-0,3796	0,3759
8	-0,1840	-0,3372	0,3537
9	-0,1301	-0,2947	0,3349
10	-0,0779	-0,1983	0,3185
11	-0,0270	-0,0996	0,3021
12	0,0228	-0,0009	0,2857
13	0,0718	0,0978	0,2693
14	0,1202	0,1471	0,2529
15	0,1681	0,1965	0,2365
16	0,2155	0,2952	0,2201
17	0,2625	0,3939	0,2536
18	0,3092	0,4926	0,3400
19	0,4019	0,5222	0,4264
20	0,5249	0,5242	0,5249

Assuming that 20 investors with different risk aversions invest an equal amount of money, the highest return would be obtained if the intraday range-based volatility estimator approach were used. This investment strategy would yield in a positive return for all risk aversions, while for portfolios from 1 to 15 it would yield in the highest portfolio return amongst all risk measures. Portfolio return for range-based volatility estimator ranges from 0,86% when diversifying risk to 0,52% when investing in only one stock, i.e. ADPL. When the investment is based on the mean-variance or lower semi-variance approach, it yields both positive and negative returns, depending on the risk aversion. Moreover, it yields lower positive returns for more diversified portfolios than the intraday range-based volatility approach. Notice that the last portfolio considers a 100% investment in a single stock, i.e. ATPL, and thereby denotes the portfolio with the highest risk. This portfolio will earn the same amount regardless of the chosen volatility estimators.

Given the mean-variance coordinates of the EF and the performance of the 1-day ex-post analysis, we conclude that the range-based volatility model outperforms both the mean-variance and the lower semi-variance models when constructing EF.

VIII. CONCLUSION

According to Markowitz, rational investors are only interested in the lower semi-variance because this estimator measures the risk of losses below a certain threshold, i.e. losses of interest to the investor. Rational investors are concerned about losses, because they want to control their portfolio risk at every point in time. When financial returns do not follow a normal distribution the standard deviation can be replaced by the lower semi-variance. Since both estimators use a single daily price observation in estimating the volatility, intraday volatility estimators can be considered as an alternative. According to [8] intraday volatility estimators are assumed to be unbiased. However, high frequency data induce microstructure effects and some practical limitation since they do not exist for all assets. The range based intraday volatility estimator has gained interest in recent literature. It is a more efficient estimator than the daily squared close-to-close return and it is relatively robust to microstructure effects. However, since there is no multivariate analogue of the range-based volatility estimator, the conditional variance-covariance matrix is estimated by a EWMA-based model, which forms the basis for the mean-variance portfolio estimation. The EFs are constructed based on all three volatility estimators. Their performances are compared in the next out-of-sample trading day.

The EF based on these three types of volatility measures show different levels of expected returns, portfolio risk and portfolio diversification. Thus, the EF differs in location on the mean-variance coordinates. The results of the three different volatility estimators show that the highest risk is found with mean-variance model and the lowest risk is measured with range-based estimator.

The efficient portfolios based on the intraday range-based volatility estimator outperforms the alternative volatility estimators for most risk levels when considering the investment in these portfolios and the returns on the next trading day.

This research shows that the choice of the risk estimator is important in constructing the EF since the portfolio weights differ and thus the choice of the investment.

For further research, we suggest to extend this theoretical research by including a longer period and to include more volatile periods like the recent credit crisis of 2007 and 2008. Intraday volatility has the interesting property of using multiple intraday observations to determine the daily volatility. According to Markowitz, rational investors are only interested in the risk of a negative return. Therefore it would be interesting to investigate the performance of a semi-variance version of the range-based volatility model on a set of financial assets.

ACKNOWLEDGMENT

This work has been fully supported by Croatian Science Foundation under the project Volatility measurement, modelling and forecasting (5199).

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