Exploring Solutions in Extended Horava-Lifshitz Gravity

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Abstract—In this letter, we explore exact solutions for the Horava-Lifshitz gravity. We use of an extension of this theory with first order dynamical lapse function. The equations of motion have been derived in a fully consistent scenario. We assume that there are some spherically symmetric families of exact solutions of this extended theory of gravity. We obtain exact solutions and investigate the singularity structures of these solutions. Specially, an exact solution with the regular horizon is found.

Keywords—Quantum gravity, Horava-Lifshitz gravity, black hole, spherically symmetric space times.

I. INTRODUCTION

IN recent years several approaches to the quantum gravity (QG) have been introduced by researchers. The aim is to solve the miracle of the quantum description of gravity in a self consistent way. By QG we mean two different meanings: one which is the most simple one is to write down the gravitational field equation with the energy momentum tensor of quantum fields. In this approach we replace the classical energy momentum tensor $T_{\mu\nu}$ by the vacuum expectation value of the energy momentum tensor of a quantum particle $< \hat{T}_{\mu\nu} >$. We can interpret gravity, in this approach, as an effect of the quantum particles on classical background. The second method is much more difficult than the first one is, how we can quantize gravity using one of the famous approaches, canonical or path integral. The main goal of both approaches is to resolve some problems in gravitational physics, especially in the high energy regime of energy, ultra-violent regime (UV). We need to regularize graviton propagator in a self consistent and reasonable way. We know that if we work in UV regime the lovely symmetry, Lorentz symmetry is not valid and we need to construct a mechanism to explain this symmetry breaking phenomena. A good way is to take space-time with different footing like $x^i \longrightarrow lx^i$, $t \longrightarrow l^z t$, where l, z, x^i and t are scaling parameter, Lifshitz parameter and space and time coordinates. It was exactly what the Horava used to propose his famous theory which we know as . Horava-Lifshitz (HL) gravity [1]–[4]. UV regime is not the only physical regime which needs modifications. Also in the low energy regime, as large scales, infra-red (IR) when general relativity needs to be modified, we need to modify the quantum scenario. It is believed that HL theory reduces to the Einstein gravity as IR limit and dark matter appears as an integration constant in

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this theory. It seems to us that modifications of HL theory are required. One possible extension is to label dynamics to lapse function, to add dynamical terms like $\{\nabla_\mu N, ..\}$ to the action [5]. This is the minimal required modification of the original HL theory. In this paper we use an extension of Horava gravity which it was proposed by several authors [7]–[10]. This extension based on the following three conditions: (i) UV completion, (ii) healthy IR behavior and (iii) a stable vacuum state in quantized version of the theory. This extended theory is renormalizable by power counting and is free of strong coupling problem. Similarly to the previously literature specially in the correspondence to the [11], we derive the full set of equations of motion of this extended Horava gravity, and then we investigate some possible classes of static spherically symmetric solutions with regular horizons.

II. GENERAL FRAME WORK OF HORAVA GRAVITY

We start with the deformed action given by

$$S = \int d^3x dt \sqrt{g} N \left(\frac{2}{k^2} K_{ij} G^{ijkl} K_{kl} - \frac{k^2}{8} E^{ij} G_{ijkl} E^{kl} + \alpha \mathcal{E}_i \mathcal{E}^i \right), \quad (1)$$

Here K_{ij} and E^{ij} are given, respectively, by

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i), \sqrt{g} E^{ij} = \frac{\delta W}{\delta g_{ij}}, \quad (2)$$

This form of action is a special case of the general form presented in [7-9], and originally was reported in [10]. We introduce:

$$W = \mu_1 \int \omega_3 + \mu_2 \int d^3x \sqrt{g}(R - 2\Lambda_W), \quad (3)$$

where

$$\omega_3 = Tr(\Gamma \wedge d\Gamma + \frac{2}{3}\Gamma \wedge \Gamma \wedge \Gamma), \quad (4)$$

Where $\mu_1(i = 1, 2)$ are coupling constants with scaling dimensions $[\mu_i]_s = i - 1$ and $[\Lambda_W]_s = 2$. The 3-vector $\mathcal{E}_i$ in (2) is given by

$$\mathcal{E}_i = \frac{\partial_i N}{N} \quad (5)$$

The action (2) can be rewritten as

$$S = -\int d^3x dt \sqrt{g} N \left(\sum_{a=1}^{6} \alpha_a \mathcal{O}_a \right), \quad (6)$$

World Academy of Science, Engineering and Technology
International Journal of Physical and Mathematical Sciences
Vol:9, No:2, 2015

where

$$\alpha_1 \equiv -\frac{2}{k^2}, \alpha_2 \equiv \frac{\alpha_4 \Lambda_W}{3\lambda - 1}, \alpha_3 = \alpha, \alpha_4 \equiv \frac{k^2 \mu_2^2}{8}, \qquad (7)$$

$$\alpha_5 \equiv -\frac{k^2 \mu_1 \mu_2}{2}, \alpha_6 \equiv \frac{k^2 \mu_1^2}{2}$$

and

$$\mathcal{O}_1 = K_{ij} G^{ijkl} K_{kl}, \mathcal{O}_2 = R - \Lambda_W, \mathcal{O}_3 = \mathcal{E}_i \mathcal{E}^i, \qquad (8)$$

$$\mathcal{O}_4 = -\frac{1-4\lambda}{4(1-3\lambda)} R^2, \mathcal{O}_5 = 0, \mathcal{O}_6 = Z_{ij} Z^{ij},$$

where μ, ω and Λ_W are constant parameters, and the symmetric tensor $Z_{ij} = C_{ij} - \frac{\mu_2}{2\mu_1} R_{ij}$ is constructed out of the Cotton tensor C_{ij} which is defined as

$$C^{ij} \equiv \epsilon^{ikl} \nabla_k \left(R^j{}_l - \frac{1}{4} R \delta_l^j \right). \qquad (9)$$

III. FIELD EQUATION

In the following we would like to consider the equations of motion for the action (6). The equations of motion obtained by varying N, N_i are respectively, given by

$$\alpha_1 \mathcal{O}_1 - \sum_{a=1}^{6} \alpha_a \mathcal{O}_a + \frac{2\alpha}{\sqrt{g}} \partial_i(\sqrt{g} \mathcal{E}^i) = 0 \qquad (10)$$

$$\nabla_j (K^{ij} - \lambda K g^{ij}) = 0 \qquad (11)$$

The equation of motion following from the variation of δg^{ij} are given by

$$\sum_{a=1}^{6} \alpha_a H_{ij}^{(a)} = 0, \qquad (12)$$

where $H_{ij}^{(a)}$ are given by

$$H_{ij}^{(1)} = \frac{N}{2}(K_{kl}K^{kl} - \lambda K^2)g_{ij} - \qquad (13)$$
$$-N(K_{ik}K^k{}_j - \lambda K K_{ij}) -$$
$$\frac{1}{\sqrt{g}} g_{ik} g_{jl}[\sqrt{g}(K^{kl}K^{kl} - \lambda K g^{kl})] -$$
$$\nabla^k[(K_{ik} - \lambda K g_{ik})N_j] -$$
$$\nabla^k[(K_{jk} - \lambda K g_{jk})N_i - (K_{ij} - \lambda K g_{ij})N_k],$$
$$H_{ij}^{(2)} = -N(R_{ij} - \frac{1}{2}K g_{ij} + \frac{3}{2}\lambda_W g_{ij}) +$$
$$(\nabla_i \nabla_j - g_{ij} \nabla_k \nabla^k)N,$$
$$H_{ij}^{(3)} = \frac{N}{2}\mathcal{E}_k \mathcal{E}^k g_{ij} - N\mathcal{E}_i \mathcal{E}_j,$$
$$H_{ij}^{(4)} = \frac{1-4\lambda}{4(1-3\lambda)}\Big[NR(2R_{ij} - \frac{1}{2}Rg_{ij}) -$$
$$2(\nabla_i \nabla_j - g_{ij}\nabla_k \nabla^k)(NR)\Big],$$
$$H_{ij}^{(5)} = \frac{1}{2}\nabla_k[\nabla_j(NZ^k{}_i) + \nabla_i(NZ^k{}_j)] -$$
$$\frac{1}{2}\nabla_k \nabla^k(NZ_{ij}) - \frac{1}{2}\nabla_k \nabla_l(NZ^{kl})g_{ij},$$
$$H_{ij}^{(6)} = \frac{1}{4}NZ_{kl}Z^{kl}g_{ij} - NZ_{ik}Z_j{}^k +$$
$$\frac{1}{2}\nabla_k[N\epsilon^{mkl}Z_{mi}R_{jl}] - \frac{1}{2}\nabla_n[NR^n{}_l\epsilon^{mkl}Z_{mi}g_{kj}]$$

$$+\frac{1}{2}\nabla_n[NZ_m{}^n \epsilon^{mkl}Z_{mi}g_{kj}R_{jl}] +$$
$$\frac{1}{2}\nabla_n \nabla^n \nabla_k[N\epsilon^{mkl}Z_{mi}g_{jl}] -$$
$$\frac{1}{2}\nabla_n[\nabla_i \nabla_k(NZ_m{}^n \epsilon^{mkl})g_{jl}] -$$
$$-\frac{1}{2}\nabla_l[\nabla_i \nabla_k(NZ_{mj}\epsilon^{mkl})] +$$
$$\frac{1}{2}\nabla_n \nabla_l \nabla_k(NZ_m{}^n \epsilon^{mkl})g_{ij} + (i \leftrightarrow j)$$

As $\alpha_3 = 0$, these equations of motion reduce to those of the original ones [6,11].

IV. STATIC, SPHERICALLY SYMMETRIC SOLUTIONS

We examine a static spherically symmetric solution with the metric ansatz given by the following:

$$ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + sin^2\theta d\varphi^2) \qquad (14)$$

It has been proven that [11], the system of field equations with $\alpha_3 = 0$ has the $(A)dS$ Schwarzshild black hole. The easiest way to obtain the solution for the full Lagrangian in the case $\alpha_3 \neq 0$, is to substitute the metric ansatz into the action, and then perform variation w.r.t the functions N and f [11]. The resulting reduced Lagrangian, up to an overal scaling constant, is given by

$$L = \frac{N}{\sqrt{f}}(2 - 3\Lambda_W r^2 - 2f - 2r\acute{f} + \frac{\lambda - 1}{2\Lambda_W}\acute{f}^2 - \qquad (15)$$

$$-\frac{2\lambda}{\Lambda_W r}(f-1)\acute{f} + \frac{2\lambda-1}{\Lambda_W r^2}(f-1)^2 + \alpha_3 f r^2 (\frac{\acute{N}}{N})^2),$$

A. Exact Solutions: Class 1

The first solution is given by

$$f = 1, N(x) = c^2(\frac{e^x}{x^2} + 2\sqrt{\frac{-3\Lambda_W}{\alpha}}\frac{c_1}{x^2} - 3\frac{c_1^2 \Lambda_W}{\alpha}e^x) \qquad (16)$$

With $\alpha \equiv \alpha_3, x = \sqrt{\frac{-3\Lambda_W}{\alpha}}r$. In brief, we examine the Lagrangian for a possible solution with $f = c.t.e$ and with an unknown auxiliary lapse function $N(r)$. If we write the associated field equations for the lapse function, we'll derive the form of (16). This solution satisfies (31). Indeed in [11], the authors showed that when f is fixed, the Newtonian potential associated with $g_{tt} = -N^2$ can be an arbitrary function of r. But now we show that we can have both f and N in this new version. Now we must examine the singularity tructure of this solution. The far field region is recovered by the last third term and in this case the metric becomes:

$$ds^2 = -e^2\sqrt{\frac{-3\Lambda_W}{\alpha}}r dt^2 + dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2) \qquad (17)$$

This is a representation of an $AdS^2 \times S^2$ spasetime. The new rescaled time coordinate is just $\bar{t} = c_1^2 c_2 \sqrt{\frac{-3\Lambda_W}{\alpha}}t$. There is a Ricci scalar singularity locates at

$$r = \sqrt{\frac{-3\Lambda_W}{\alpha}}log(c_1\sqrt{\frac{-3\Lambda_W}{\alpha}}) \qquad (18)$$

The solution is asymtotically AdS^4. We fix it as the following:

$$c_1\sqrt{\frac{-3\Lambda_W}{\alpha}} = 1 \qquad (19)$$

Thus the singularity is located at $r = 0$ wich is a coordinate singularity and not a naked ones. In this case, the metric bocomes like the following form:

$$ds^2 = -(\frac{e^x}{x^2}+\frac{2}{x}-e^x)^2dt^2+c_1^2dr^2+r^2(d\theta^2+sin^2\theta d\varphi^2) \quad (20)$$

with a suitable coordinates definition $\bar{t} = c_2t$. The Newtonian potential $g_{tt} = -N(x)^2$ has a maxima which is located at $(x = 0, g_{tt} = 1.25341)$,thus it means that we have a maximum for the redshift, when the signal is emitted from origin $r = 0$. It is suitable for us to invesigate the behavior of the gravitational redshift. Using the famous formula of the gravitational redshift in an asymptotically flat spasetime, for the (20) spase (which is asymptotically Ads) we have

$$1 + z = \sqrt{\frac{g_{tt}(x \longrightarrow \infty)}{g_{tt}(x)}} \qquad (21)$$

The divergency in the value of the redshift can be addressed as the existence of a horizon. This horizon, as a null and hypersurface orthogonal surface, is located in the vicinity of the point $x = 1.246$.

B. Case with $\lambda = 1$

It is remarkable to invesigate the case with $\lambda = 1$ solution. This case corresponds to the IR regime. In this case, the functions f and N are given by

$$N^2 = f = (\frac{c_1}{r} - c_2)^2 \qquad (22)$$

The solution is assymptotically flat, with a horizon at $r = h$, where h is the root of f which is located at

$$r = h = \frac{c_1}{c_2} \qquad (23)$$

The exact solution in this case can be written as

$$ds^2 = (\frac{c_1}{r} - c_2)^2(-dt^2+dr^2)+r^2(d\theta^2+sin^2(\theta)d\varphi^2) \quad (24)$$

As we observe now, this solution is asymptotically flat. Thus using the usual formula for the gravitational redshift in an asymptotically flat spasetime we have:

$$1 + z = \sqrt{\frac{g_{tt}(\infty)}{g_{tt}(r)}} = \left|\frac{x}{c_1 - x}\right| \qquad (25)$$

Where $x = c_2r$. The singularity is located $x = c_1$. To recover the flat spasetime metric as $r \longrightarrow \infty$ we need to impose r=the condition $c_2 = 1$. The unknown constant $c_1 = a$ can be interpreted as the mass of the spasetime. As we can see, the redshift for this model, reaches to the maximum values for severel times.

C. Gase with $\Lambda_W = 0$

In this case both f and N are determined, given by

$$N = r^\beta\sqrt{f}, f = r^{\gamma_\pm} \qquad (26)$$

$$\gamma_\pm = \frac{-\beta\lambda + \beta + \lambda - 1}{2(\lambda - 1)} \pm \qquad (27)$$
$$\pm(\frac{\sqrt{\beta^2\lambda^2 - 2\beta^2\lambda + 6\beta\lambda^2 - 4\beta\lambda + \beta^2}}{2(\lambda - 1)} -$$
$$-\frac{\sqrt{2\beta + 9\lambda^2 - 18\lambda + 9}}{2(\lambda - 1)}),$$

where β is another integration constant. If we want that the solution to be real, it is necessary and sufficent to have $\lambda > 1$. This solution has a curvature singularity at $r = 0$ for general λ in agreement with the results of [11]. It also has a curvature singularity at $x = \infty$ if $\beta > 0$. The exact solution in this case can be written as

$$ds^2 = -r^{2\beta+\gamma_\pm}dt^2+r^{-\gamma_\pm}dr^2+r^2(d\theta^2+sin^2(\theta)d\varphi^2) \quad (28)$$

For some values of the pair $\beta, \gamma_\pm$ that $\gamma_\pm = -\beta$ the solution (28) converts to a conformally stationary solution

$$ds^2 = e^{2U}(-dt^2+dr^2)+r^2(d\theta^2+sin^2(\theta)d\varphi^2) \quad (29)$$

Where the Newtonian potential $U(r) = \frac{\beta}{2}log(r)$. Indeed there exist a vast family of such exact solutions which can be written as the below

$$ds^2 = e^{2U(\xi)}(-dt^2+d\xi^2)+e^{2K(\xi)}(d\theta^2+sin^2(\theta)d\varphi^2) \quad (30)$$

Here $\xi = \frac{r^{1-\beta-\gamma_\pm}}{1-\beta-\gamma_\pm}$

$$e^{U(\xi)} = ((1-\beta-\gamma_\pm)\xi)^{\frac{\beta+\gamma_\pm}{2(1-\beta-\gamma_\pm)}} \qquad (31)$$

$$e^{K(\xi)} = ((1-\beta-\gamma_\pm)\xi)^{\frac{1}{1-\beta-\gamma_\pm}} \qquad (32)$$

In the last one, the potential functions are defined for $\beta < \frac{1}{2}, \gamma_\pm = \beta$

V. CONCLUSION

Horava-Lifshitz gravity is a potentially powerful alternative candidate for quatum gravity. In its original form, it must be modified to give the correct cosmological and dynamical predictions. One of the important problems in the old version of this theory was why the non dynamical lapse function appeared?. This may be solved by inserting a new auxillary field in the action and investigation of it's dynamical behavior. In this work, motivated by some modifications of this theory, we study the spherically symmetric solutions in this version. We obtained three different class of solutions.

ACKNOWLEDGMENT

REFERENCES

[1] P.Horava, *Quantum Criticality and Yang-Mills Gauge Theory*. Phys.Lett. B694 (2010) 172-176. DOI: 10.1016/j.physletb.2010.09.055

[2] P.Horava, *Membranes at Quantum Criticality*. JHEP, 03(2009)020 doi:10.1088/1126-6708/2009/03/020

[3] P.Horava, *Quantum Gravity at a Lifshitz Point*. Phys.Rew.D79 (2009) 084008. DOI: 10.1103/PhysRevD.79.084008

[4] P.Horava, *Spectral Dimension of the Universe in Quantum Gravity at a Lifshitz Pointar*. Phys.Rev.Lett. 102 (2009) 161301 DOI: 10.1103/PhysRevLett.102.161301.

[5] D.Blas, O.Pujolas, S.Sibiryakov, *On the Extra Mode and Inconsistency of Horava Gravity*. JHEP10(2009)029 doi:10.1088/1126-6708/2009/10/029

[6] E.Kiritsis and G. Kofinas, *Hoava-Lifshitz cosmology*. Nucl.Phys.B821:467-480,2009.

[7] Elias Kiritsis, *Spherically symmetric solutions in modified Horava-Lifshitz gravity*. Phys.Rev. D81 (2010) 044009. DOI: 10.1103/PhysRevD.81.044009. CCTP-2009-20

[8] D.Blas, O.Pujolas, S.Sibiryakov, *Consistent Extension of Horava Gravity*. Phys.Rev.Lett. 104 (2010) 181302. DOI: 10.1103/PhysRevLett.104.181302. CERN-PH-TH-2009-173

[9] D.Blas, O.Pujolas, S.Sibiryakov, *Comment on 'Strong coupling in extended Horava-Lifshitz gravity'*. Phys.Lett.B688:350-355,2010.

[10] F.-W.Shu, *Extended Horava Gravity with Physical Groumd-State Wavefunction.* arXiv:1009.3677 [hep-th].

[11] H.Lu, Jianwei Mei and C.N. Pope, *Solutions to Horava Gravity*. Phys. Rev. Lett. 103, 091301-2009

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