World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:8, No:10, 2014

Operational Representation of Certain Hypergeometric Functions by Means of Fractional Derivatives and Integrals

Manoj Singh, Mumtaz Ahmad Khan, Abdul Hakim Khan

Abstract—The investigation in the present paper is to obtain certain types of relations for the well known hypergeometric functions by employing the technique of fractional derivative and integral.

Mathematics Subject Classification(2010): Primary 42C05, Secondary 33C20, 26A33.

Keywords—Fractional Derivatives and Integrals, Hypergeometric functions.

I. Introduction

THE names fractional calculus is concerned with the generalization of differentiation and integration to fractional order. There are number of ways of defining fractional derivatives and integrals.

We begin by recalling S.F. Lacroix [9] definition of mth derivatives for $y = x^n$, where n is a positive integer as

$$\frac{d^m y}{dx^n} = \frac{(m!)}{(n-m)!} x^{n-m} \tag{1}$$

A vital development in the field of fractional calculus was laid down by L. Euler, N.H. Abel, J. Liouville, and B. Riemann.

For the function $f(x)$, expanded in series form

$$f(x) = \sum_{n=0}^{\infty} c_n \exp(a_n x) \tag{2}$$

Liouville defined the fractional derivative of order ν by

$$D_x^{\nu}\{f(x)\} = \sum_{n=0}^{\infty} c_n a_n^{\nu} \exp(a_n x) \tag{3}$$

In 1931, Euler extended the derivative formula

$$D_z^n\{z^{\lambda}\} = \lambda(\lambda-1), ...(\lambda-n+1) z^{\lambda-n}, \quad (n = 0, 1, 2, ...)$$
$$= \frac{\Gamma(\lambda+1)}{(\lambda-n+1)} z^{\lambda-n} \tag{4}$$

In the last few years fractional calculus became one of the most intensively developing areas of mathematical analysis. Its field of applications is in almost every field of science, engineering, and mathematics. Several applications of fractional calculus are in fluid dynamics, stochastic dynamical system, astrophysics, probability theory and statistics, image processing, nonlinear control theory, plasma physics etcetera.

Some of the definition and notations used in this paper are as follows:

Appell's gave four hypergeometric functions of two variables given by [6].

$$F_1[a, b, b';\ c;\ x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(c)_{m+n}\ m!\ n!}, \tag{5}$$

$$F_2[a, b, b';\ c, c';\ x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(c)_m\ (c')_n\ m!\ n!}, \tag{6}$$

$$F_3[a, a', b, b';\ c;\ x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n x^m y^n}{(c)_{m+n}\ m!\ n!}, \tag{7}$$

$$F_4[a, b;\ c, c';\ x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n} x^m y^n}{(c)_m(c')_n\ m!\ n!}. \tag{8}$$

In 2002, M.A. Khan and G.S. Abukhammash [5] generalized the Appell's functions of two variables and introduce 10 Appell's type generalized functions M_i, $i = 1, 2, ..., 10$ by considering the product of two ${}_3F_2$ function, but we use only seven hypergeometric functions as mentioned below:

$$M_1\,(a, a', b, b', c, c'; d, e, e'; x, y)$$
$$= \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n(c)_m(c')_n}{(d)_{m+n}(e)_m(e')_n} \frac{x^m y^n}{m!\ n!} \tag{9}$$

$$M_2\,(a, a', b, b', c, c'; d, e; x, y)$$
$$= \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n(c)_m(c')_n}{(d)_{m+n}(e)_{m+n}} \frac{x^m y^n}{m!\ n!} \tag{10}$$

$$M_3\,(a, b, b', c, c'; d, d', e, e'; x, y)$$
$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n(c)_m(c')_n}{(d)_m(d')_n(e)_m(e')_n} \frac{x^m y^n}{m!\ n!} \tag{11}$$

$$M_4\,(a, b, b', c, c'; d, e, e'; x, y)$$
$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n(c)_m(c')_n}{(d)_{m+n}(e)_m(e')_n} \frac{x^m y^n}{m!\ n!} \tag{12}$$

$$M_7\,(a, b, c, c'; d, e; x, y)$$

Singh M. is with Department of Mathematics, College of Science, Jazan University, Jazan, Saudi Arabia (e-mail: manojsingh221181@gmail.com).

Khan M. A. is with Department of Applied Mathematics, Faculty of Engineering, Aligarh Muslim University, Aligarh - 202002, U.P., India (e-mail: mumtaz_ahmad_khan_2008@yahoo.com).

Khan A. H. is with Department of Applied Mathematics, Faculty of Engineering, Aligarh Muslim University, Aligarh - 202002, U.P., India (e-mail: ahkhanamu@gmail.com).

$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_m(c')_n}{(d)_{m+n}(e)_m(e')_n} \frac{x^m y^n}{m!\ n!} \tag{13}$$

$$M_8\,(a,b,c,c';d,e,e';x,y)$$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_m(c')_n}{(d)_{m+n}(e)_{m+n}} \frac{x^m y^n}{m!\ n!} \tag{14}$$

$$M_{10}\,(a,b,c;d,e,e';x,y)$$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(c)_{m+n}}{(d)_{m+n}(e)_m(e')_n} \frac{x^m y^n}{m!\ n!} \tag{15}$$

Lauricella [1], generalized the Appell double hypergeometric functions $F_1,...,F_4$ to functions of n variables, but we use only two $F_A^{(n)}$ and $F_D^{(n)}$ are defined by

$$F_A^{(n)}\,[a,b_1,...,b_n;\ c_1,...,c_n;\ x_1,...,x_n]$$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a)_{m_1+...+m_n}\ (b_1)_{m_1}...(b_n)_{m_n}}{(c_1)_{m_1}......(c_n)_{m_n}} \times \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!} \tag{16}$$

$$F_D^{(n)}\,[a,b_1,...,b_n;\ c\ ;\ x_1,...,x_n]$$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a)_{m_1+...+m_n}\ (b_1)_{m_1}...(b_n)_{m_n}}{(c)_{m_1+...+m_n}} \times \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!} \tag{17}$$

In 1963, Pandey [7] established two interesting Horn's type hypergeometric functions of three variables while transforming Pochhammer's double-loop contour integrals associated with the Lauricella's functions F_G and F_F as given below:

$$G_A\,[\alpha,\beta,\beta';\gamma;x,y,z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}\ (\beta)_{m+p}\ (\beta')_n}{(\gamma)_{n+p-m}\ m!\ n!\ p!} x^m\ y^n\ z^p \tag{18}$$

$$G_B\,[\alpha,\beta_1,\beta_2,\beta_3;\gamma;x,y,z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\gamma)_{n+p-m}\ m!\ n!\ p!} x^m\ y^n\ z^p \tag{19}$$

II. Fractional Derivative Relations

In this section we derive certain types of fractional derivative relations are as follows:

$$D_{x_1}^{\mu-\alpha} D_{x_2}^{\mu'-\alpha'} \left\{ {x_1}^{-\alpha} {x_2}^{-\alpha'} \left(1-\frac{\omega_1}{x_1}-\frac{\omega_2}{x_2}\right)^{-\beta} \right\}$$

$$= \frac{\Gamma(1-\alpha)\Gamma(1-\alpha')}{\Gamma(1-\mu)\Gamma(1-\mu')} {x_1}^{-\mu} {x_2}^{-\mu'} \times F_2\left[\beta,\mu,\mu';\alpha,\alpha';\frac{\omega_1}{x_1},\frac{\omega_2}{x_2}\right] \tag{20}$$

where, $\left|\frac{\omega_1}{x_1}+\frac{\omega_2}{x_2}\right|<1.$

$$D_{x_1}^{\mu_1-\alpha_1}...D_{x_n}^{\mu_n-\alpha_n} \left\{ {x_1}^{-\alpha_1}...{x_n}^{-\alpha_n} \left(1-\frac{\omega_1}{x_1}-...-\frac{\omega_n}{x_n}\right)^{-\beta} \right\}$$

$$= \prod_{j=1}^{n} \left\{ \frac{\Gamma(1-\alpha_j)}{\Gamma(1-\mu_j)} (x_j)^{-\mu_j} \right\} \times F_A^{(n)}\left[\beta,\mu_1,...,\mu_n;\alpha_1,...\alpha_n;\frac{\omega_1}{x_1},...,\frac{\omega_n}{x_n}\right] \tag{21}$$

where, $\left|\frac{\omega_1}{x_1}+.....+\frac{\omega_n}{x_n}\right|<1.$

$$D_{x_1}^{\mu_1-\alpha_1} D_{x_2}^{\mu_2-\alpha_2} D_{x_3}^{\mu_3-\alpha_3} D_{x_4}^{\mu_4-\alpha_4} \times \left\{ {x_1}^{-\alpha_1}{x_2}^{-\alpha_2}{x_3}^{-\alpha_3}{x_4}^{-\alpha_4} \left(1-\frac{\omega_1}{x_1x_2}-\frac{\omega_2}{x_3x_4}\right)^{-\beta} \right\}$$

$$= \frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(1-\alpha_3)\Gamma(1-\alpha_4)}{\Gamma(1-\mu_1)\Gamma(1-\mu_2)\Gamma(1-\mu_3)\Gamma(1-\mu_4)} \times {x_1}^{-\mu_1}{x_2}^{-\mu_2}{x_3}^{-\mu_3}{x_4}^{-\mu_4} \times M_3\left[\beta,\mu_1,\mu_2,\mu_3,\mu_4;\alpha_1,\alpha_2,\alpha_3,\alpha_4;\frac{\omega_1}{x_1x_2},\frac{\omega_2}{x_3x_4}\right] \tag{22}$$

where, $\left|\frac{\omega_1}{x_1x_2}+\frac{\omega_2}{x_3x_4}\right|<1.$

$$D_{x_1}^{\mu_1-\alpha_1} D_{x_2}^{\mu_2-\alpha_2} D_{x_3}^{\mu_3-\alpha_3} \times \left\{ {x_1}^{-\alpha_1}{x_2}^{-\alpha_2}{x_3}^{-\alpha_3} \left(1-\frac{\omega_1}{x_1x_2}-\frac{\omega_2}{x_1x_3}\right)^{-\beta} \right\}$$

$$= \frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(1-\alpha_3)}{\Gamma(1-\mu_1)\Gamma(1-\mu_2)\Gamma(1-\mu_3)} {x_1}^{-\mu_1}{x_2}^{-\mu_2}{x_3}^{-\mu_3} \times M_7\left[\beta,\mu_1,\mu_2,\mu_3;\alpha_1,\alpha_2,\alpha_3;\frac{\omega_1}{x_1x_2},\frac{\omega_2}{x_1x_3}\right] \tag{23}$$

where, $\left|\frac{\omega_1}{x_1x_2}+\frac{\omega_2}{x_1x_3}\right|<1.$

$$D_x^{\alpha-\mu} \left\{ x^{\alpha}(1-x)^{-\beta} \left(1-\omega_1 x-\frac{\omega_2}{1-x}\right)^{-\gamma} \right\}$$

$$= \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)} x^{\mu} H_A\left[\beta,\gamma,1+\alpha;\beta,1+\mu;\omega_2,\omega_1 x,x\right] \tag{24}$$

where, $Re(\alpha)\geq 0,\ |x|<1,\ \left|\omega_1 x+\frac{\omega_2}{1-x}\right|<1.$

$$D_x^{\alpha-\mu} \left\{ x^{\alpha}(1-x)^{-\beta} \left(1-\frac{\omega x}{1-x}\right)^{-\gamma} \right\}$$

$$= \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)} x^{\mu}(1-x)^{-\alpha-1} \times F_1\left[1+\alpha,\gamma,1+\mu-\beta;1+\mu;\frac{\omega x}{1-x},\frac{-x}{1-x}\right] \tag{25}$$

where, $Re(\alpha)\geq 0,\ |x|<1,\ \left|\frac{\omega x}{1-x}\right|<1.$

$$D_x^{\alpha-\mu} \left\{ x^{\alpha}(1-x)^{-\beta} \left(1-\frac{\omega_1 x}{1-x}\right)^{-\gamma} \left(1-\frac{\omega_2 x}{1-x}\right)^{-\delta} \right\}$$

$$= \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)} x^{\mu}(1-x)^{-\alpha-1}\, F_D^{(3)}\left[1+\alpha,\ \gamma,\ \delta,\ 1+\mu-\beta;\ 1+\mu;\ \frac{\omega_1 x}{1-x}, \frac{\omega_2 x}{1-x}, \frac{x}{x-1}\right] \quad (26)$$

where, $Re(\alpha) \geq 0,\ |x| < 1,\ \left|\frac{\omega_1 x}{1-x}\right| < 1,\ \left|\frac{\omega_2 x}{1-x}\right| < 1$ and $F_D^{(3)}$ is defined by eq. (17) at $n = 3$

$$D_x^{\alpha-\mu}\left\{x^{\alpha}(1-x)^{-\beta}(1-\omega_1 x)^{-\gamma}\left(1-\frac{\omega_2}{1-x}\right)^{-\delta}\right\}$$

$$= \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)}\ x^{\mu}\ F_M\left[\delta,\ 1+\alpha,\ 1+\alpha,\ \beta,\ \gamma,\ \beta\ ;\beta, 1+\mu, 1+\mu; \omega_2, \omega_1 x, x\right] \quad (27)$$

where, $Re(\alpha) \geq 0,\ |x| < 1,\ |\omega_1 x| < 1,\ \left|\frac{\omega_2}{1-x}\right| < 1$ and F_m is defined by saran [8]

$$F_M\left[\alpha_1,\ \alpha_2,\ \alpha_2,\ \beta_1,\ \beta_2,\ \beta_1;\ \gamma_1,\ \gamma_2,\ \gamma_2;\ x, y, z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m\ (\alpha_2)_{n+p}\ (\beta_1)_{m+p}\ (\beta_2)_n}{(\gamma_1)_m\ (\gamma_2)_{n+p}\ m!\ n!\ p!} x^m\ y^n\ z^p \quad (28)$$

$$D_x^{\alpha-\mu}\left\{x^{\alpha}(1-x)^{-\beta}\,(1-\omega_1 x)^{-\gamma}\left(1-\frac{\omega_2 x}{1-x}\right)^{-\delta}\right\}$$

$$= \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)} x^{\mu} F^{(3)}\left[\begin{matrix} 1+\alpha\ :: \text{---}\ ;\ \beta\ ;\text{---} \\ 1+\mu\ :: \text{---}\ ;\text{---}\ ;\text{---} \end{matrix}\ \begin{matrix} ::\ \gamma\ ;\ \delta\ ;\text{---}\ ; \\ :: \text{---}\ ;\ \beta\ ;\text{---}\ ; \end{matrix}\ \omega_1 x, \omega_2 x, x\right] \quad (29)$$

where, $Re(\alpha) \geq 0,\ |x| < 1,\ |\omega_1 x| < 1,\ \left|\frac{\omega_2 x}{1-x}\right| < 1$ and $F^{(3)}[x, y, z]$ is triple hypergeometric series defined by Srivastava (see [4], p. 428).

$$D_{x_1}^{\mu_1-\alpha_1} D_{x_2}^{\mu_2-\alpha_2} D_{x_3}^{\mu_3-\alpha_3} \times \left\{{x_1}^{-\alpha_1}{x_2}^{-\alpha_2}{x_3}^{-\alpha_3}\left(1-\frac{\omega_1}{x_1 x_2}\right)^{-\beta}\left(1-\frac{\omega_2}{x_1 x_3}\right)^{-\gamma}\right\}$$

$$= \frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(1-\alpha_3)}{\Gamma(1-\mu_1)\Gamma(1-\mu_2)\Gamma(1-\mu_3)} {x_1}^{-\mu_1}{x_2}^{-\mu_2}{x_3}^{-\mu_3} \times M_4\left[\mu_1, \beta, \gamma, \mu_2, \mu_3; \alpha_1, \alpha_2, \alpha_3; \frac{\omega_1}{x_1 x_2}, \frac{\omega_2}{x_1 x_3}\right] \quad (30)$$

where, $\left|\frac{\omega_1}{x_1 x_2}\right| < 1,\ \left|\frac{\omega_2}{x_1 x_3}\right| < 1.$

$$D_{x_1}^{\alpha-\mu} D_{x_2}^{\alpha'-\mu'}\left\{{x_1}^{\alpha}{x_2}^{\alpha'}(1-\omega_1 x_1 x_2)^{-\beta}(1-\omega_2 x_1 x_2)^{-\gamma}\right\}$$

$$= \frac{\Gamma(1+\alpha)\Gamma(1+\alpha')}{\Gamma(1+\mu)\Gamma(1+\mu')} {x_1}^{\mu}{x_2}^{\mu} \times M_8\left[1+\alpha, 1+\alpha', \beta, \gamma; 1+\mu, 1+\mu'; \omega_1 x_1 x_2, \omega_2 x_1 x_2\right] \quad (31)$$

where, $Re(\alpha) \geq 0,\ Re(\alpha') \geq 0,\ |\omega_1 x_1 x_2| < 1,\ |\omega_2 x_1 x_2| < 1.$

$$D_x^{\alpha-\mu}\left\{x^{\alpha}(1-\omega_1 x)^{-\beta}\left(1-\omega_2 x-\frac{\omega_3}{x}\right)^{-\gamma}\right\}$$

$$= \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)} x^{\mu} G_A\left[1+\alpha, \gamma, \beta; 1+\mu; \omega_1 x, \omega_2 x, \frac{\omega_3}{x}\right] \quad (32)$$

where, $Re(\alpha) \geq 0,\ |\omega_1 x| < 1,\ \left|\omega_2 x + \frac{\omega_3}{x}\right| < 1.$

$$D_x^{\mu-\alpha}\left\{x^{-\alpha}\left(1-\frac{\omega_1}{x}\right)^{-\beta}\left(1-\frac{\omega_2}{x}\right)^{-\gamma}\right\}$$

$$= \frac{\Gamma(1-\alpha)}{\Gamma(1-\mu)} x^{-\mu} F_1\left[\mu,\ \beta,\ \gamma;\ \alpha; \frac{\omega_1}{x}, \frac{\omega_2}{x}\right] \quad (33)$$

where, $\left|\frac{\omega_1}{x}\right| < 1,\ \left|\frac{\omega_2}{x}\right| < 1.$

$$D_x^{\alpha-\mu}\left\{x^{\alpha}(1-\omega_1 x)^{-\beta}(1-\omega_2 x)^{-\gamma}\left(1-\frac{\omega_3}{x}\right)^{-\delta}\right\}$$

$$= \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)} x^{\mu} G_B\left[1+\alpha, \delta, \beta, \gamma; 1+\mu; \frac{\omega_3}{x}, \omega_1 x, \omega_2 x\right] \quad (34)$$

where, $Re(\alpha) \geq 0,\ |\omega_1 x| < 1,\ |\omega_2 x| < 1,\ \left|\frac{\omega_3}{x}\right| < 1.$

$$D_x^{\alpha-\mu}\left\{x^{\alpha}(1-\omega_1 x)^{-\beta}(1-\omega_2 x)^{-\gamma}(1-\omega_3 x)^{-\delta}\right\}$$

$$= \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu)} x^{\mu} F_D^{(3)}\left[1+\alpha, \beta, \gamma, \delta; 1+\mu; \omega_1 x, \omega_2 x, \omega_3 x\right] \quad (35)$$

where, $Re(\alpha) \geq 0,\ |\omega_1 x| < 1,\ |\omega_2 x| < 1,\ |\omega_3 x| < 1.$

$$D_x^{\mu-\alpha}\left\{x^{-\alpha}\left(1-\frac{\omega_1}{x}\right)^{-\beta_1}\left(1-\frac{\omega_2}{x}\right)^{-\beta_2}...\left(1-\frac{\omega_n}{x}\right)^{-\beta_n}\right\}$$

$$= \frac{\Gamma(1-\alpha)}{\Gamma(1-\mu)} x^{-\mu} F_D^{(n)}\left[\mu, \beta_1, \beta_2, ..., \beta_n; \alpha; \frac{\omega_1}{x}, \frac{\omega_2}{x}, ..., \frac{\omega_n}{x}\right] \quad (36)$$

where, $\left|\frac{\omega_1}{x}\right| < 1,\ \left|\frac{\omega_2}{x}\right| < 1, ...,\ \left|\frac{\omega_n}{x}\right| < 1.$

$$D_x^{\mu-\alpha}\left\{x^{-\alpha}\left(1-\frac{1}{x}\right)^{-\beta}\left(1-\frac{\omega_1}{x}\right)^{-\gamma_1}...\left(1-\frac{\omega_n}{x}\right)^{-\gamma_n}\right\}$$

$$= \frac{\Gamma(1-\alpha)}{\Gamma(1-\mu)} x^{\beta-\mu}(x-1)^{-\beta} \times {}^{(1)}_{(2)}E_D^{(n+1)}\left[\beta, \mu, \alpha-\mu, \gamma_1, ..., \gamma_n; \alpha; \frac{\omega_1}{x}, ..., \frac{\omega_n}{x}, \frac{1}{1-x}\right] \quad (37)$$

where, $\left|\frac{1}{x}\right| < 1,\ \left|\frac{\omega_1}{x}\right| < 1, ...,\ \left|\frac{\omega_n}{x}\right| < 1$ and ${}^{(1)}_{(2)}E_D^{(n+1)}$ is defined by Exton ([2], p.89) at $k = 1$ and $n = n + 1$ as

$${}^{(k)}_{(2)}E_D^{(n)}\left[a, a', b_1, ..., b_n; c; x_1, ..., x_n\right]$$

$$= \sum_{m_1,...,m_n}^{\infty} \frac{(a)_{m_1+...+m_k}(a')_{m_{k+1}+...+m_n}(b_1)_{m_1}...(b_n)_{m_n}}{(c)_{m_1+...+m_k}} \times \frac{x_1^{m_1}...\ x_n^{m_n}}{m_1!....m_n!} \quad (38)$$

III. Fractional Integration Relations

The rule for fractional integration by part is in the form

$$\int_a^b D^{\nu}_{(a,x)}\{f(x)\}g(x)dx = \int_a^b f(x)D^{\nu}_{(x,b)}\{g(x)\}dx. \quad (39)$$

where $D^{\nu}_{(a,x)}$ and $D^{\nu}_{(x,b)}$ denotes the operators of fractional derivatives (of order ν) which can be defined, if $Re(\nu) < 0$, by the following integrals

$$D^{\nu}_{(a,x)}\{f(x)\} = \frac{1}{\Gamma(-\nu)}\int_a^x (x-t)^{-\nu-1}f(t)dt, \quad (40)$$

and

$$D^{\nu}_{(x,b)}\{g(x)\} = \frac{1}{\Gamma(-\nu)}\int_x^b (t-x)^{-\nu-1}g(t)dt, \quad (41)$$

If $f(x)$ and $g(x)$ are functions defined by

$$f(x) = \sum_{r=0}^{\infty} A_r (x-a)^{\rho+n-1},$$

$$g(x) = \sum_{s=0}^{\infty} B_s (b-x)^{\sigma+n-1}. \quad (42)$$

then the fractional derivatives are obtained by differentiating these series term-by-term and applying the definition (1).

In this section, certain forms of integrals have been found by adopting the technique defined by (39) for different types of hypergeometric functions as mentioned below:

$$F_1[\alpha,\beta,\beta';\gamma;x,y] = \frac{\Gamma(\gamma)}{\Gamma(\alpha')\Gamma(\gamma-\alpha')}$$

$$\times \int_0^1 u^{\alpha'-1}(1-u)^{\gamma-\alpha'-1}F_1[\alpha,\beta,\beta';\alpha';xu,yu]\,du \quad (43)$$

$Re(\gamma) > Re(\alpha') > 0,\ |x| < 1,\ |y| < 1.$

Proof: By using ([3], p. 276, eq. 7)

$$F_1[\alpha,\beta,\beta';\gamma;x,y] = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}$$

$$\times \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}du \quad (44)$$

with the aid of (1), we see that

$$\frac{d^{\alpha-\alpha'}}{d(1-u)^{\alpha-\alpha'}}(1-u)^{\gamma-\alpha'-1} = \frac{\Gamma(\gamma-\alpha')}{\Gamma(\gamma-\alpha)}(1-u)^{\gamma-\alpha-1}$$

Substituting in the last integral and using the fractional integration by part formula (39), one obtains

$$F_1[\alpha,\beta,\beta';\gamma;x,y] = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha')}\int_0^1 (1-u)^{\gamma-\alpha'-1}$$

$$\times \frac{d^{\alpha-\alpha'}}{d(u)^{\alpha-\alpha'}}\{u^{\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}\}du \quad (45)$$

Now, the use of binomial theorem and (1), gives

$$\frac{d^{\alpha-\alpha'}}{d(u)^{\alpha-\alpha'}}\{u^{\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}\}$$

$$= \sum_{r,s=0}^{\infty} \frac{(\beta)_r(\beta')_s}{r!\ s!}\frac{\Gamma(\alpha+r+s)}{\Gamma(\alpha'+r+s)}(u)^{\alpha'+r+s-1}$$

using the above relation in the last integral (45), we get the required result.

Next, we obtain some more integral representation for the Appell series F_3 defined by (7), by using fractional integration by part as given below:

$$F_3[\alpha,\alpha',\beta,\beta';\alpha+\alpha';x,y]$$

$$= \frac{\Gamma(\alpha+\alpha')}{\Gamma(\alpha+\alpha'-\beta)\Gamma(\beta)}\int_0^1 u^{\beta-1}(1-ux)^{-\alpha}(1-u)^{\alpha+\alpha'-\beta-1}$$

$$\times {}_2F_1\left[\begin{matrix}\alpha',\ \beta\ ;\\ \alpha+\alpha'-\beta;\end{matrix}\ (1-u)y\right]du \quad (46)$$

$Re(\beta) > 0,\ Re(\alpha+\alpha') > Re(\beta),\ |x| < 1,\ |y| < 1.$

Alternatively, equivalently

$$F_3[\alpha,\alpha',\beta,\beta';\beta+\beta';x,y]$$

$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta+\beta'-\alpha)\Gamma(\alpha)}\int_0^1 u^{\alpha-1}(1-ux)^{-\beta}(1-u)^{\beta+\beta'-\alpha-1}$$

$$\times {}_2F_1\left[\begin{matrix}\beta',\ \alpha'\ ;\\ \beta+\beta'-\alpha;\end{matrix}\ (1-u)y\right]du \quad (47)$$

$Re(\alpha) > 0,\ Re(\beta+\beta') > Re(\alpha),\ |x| < 1,\ |y| < 1.$

Also,

$$F_3[\alpha,\alpha',\beta,\beta';\rho+\rho';x,y]$$

$$= \frac{\Gamma(\rho+\rho')}{\Gamma(\rho+\rho'-\mu)\Gamma(\mu)}\int_0^1 u^{\mu-1}(1-u)^{\rho+\rho'-\mu-1}$$

$$\times {}_2F_1\left[\begin{matrix}\alpha,\beta;\\ \mu\ ;\end{matrix}\ ux\right] {}_2F_1\left[\begin{matrix}\alpha',\ \beta'\ ;\\ \rho+\rho'-\mu;\end{matrix}\ (1-u)y\right]du \quad (48)$$

$Re(\mu) > 0,\ Re(\rho+\rho') > Re(\mu),\ |x| < 1,\ |y| < 1.$

Proof of (46): Applying the integral relation ([3], p. 279, eq. 17),

$$F_3[\alpha,\alpha',\beta,\beta';\alpha+\alpha';x,y]$$

$$= \frac{\Gamma(\alpha+\alpha')}{\Gamma(\alpha)\Gamma(\alpha')}\int_0^1 u^{\alpha-1}(1-u)^{\alpha'-1}(1-ux)^{-\beta}$$

$$\times(1-(1-u)y)^{-\beta'}du \quad (49)$$

Now, by using (1), we see that

$$\frac{u^{\alpha-1}}{\Gamma(\alpha)}(1-ux)^{-\beta} = \frac{d^{\beta-\alpha}}{d(u)^{\beta-\alpha}}\left\{\frac{u^{\beta-1}}{\Gamma(\beta)}(1-ux)^{-\alpha}\right\}$$

Substituting this in last integral (49), and by using the formula (39), we get

$$F_3\left[\alpha,\alpha',\beta,\beta';\alpha+\alpha';x,y\right]$$

$$= \frac{\Gamma(\alpha+\alpha')}{\Gamma(\beta)}\int_0^1 u^{\beta-1}(1-ux)^{-\alpha}$$

$$\times\frac{d^{\beta-\alpha}}{d(1-u)^{\beta-\alpha}}\left\{\frac{(1-u)^{\alpha'-1}}{\Gamma(\alpha')}(1-(1-u)y)^{-\beta'}\right\}du \quad (50)$$

Further, by using the binomial theorem and (1), we obtain the required result (46).

The proof of integral relation (47) is similar to the proof of (46).

Proof of (48): Making the use of ([3], p. 279, Eq. 16),

$$F_3\left[\alpha,\alpha',\beta,\beta';\rho+\rho';x,y\right] = \frac{\Gamma(\rho+\rho')}{\Gamma(\rho)\Gamma(\rho')}\int_0^1 u^{\rho-1}(1-u)^{\rho'-1}$$

$$\times\,{}_2F_1\left[\begin{matrix}\alpha,\beta; \\ \rho\;;\end{matrix}\; ux\right]{}_2F_1\left[\begin{matrix}\alpha',\beta'; \\ \rho'\;;\end{matrix}\; (1-u)y\right]du \quad (51)$$

Now, by using (1) and the series representation ${}_2F_1$, it follows that

$$\frac{u^{\rho-1}}{\Gamma(\rho)}\;{}_2F_1\left[\begin{matrix}\alpha,\beta; \\ \rho\;;\end{matrix}\; ux\right] = \frac{d^{\mu-\rho}}{d(u)^{\mu-\rho}}\left\{\frac{u^{\mu-1}}{\Gamma(\mu)}\;{}_2F_1\left[\begin{matrix}\alpha,\beta; \\ \mu\;;\end{matrix}\; ux\right]\right\}$$

Substituting this in last integral (51) and by using the formula (39), we get

$$F_3\left[\alpha,\alpha',\beta,\beta';\rho+\rho';x,y\right]$$

$$= \frac{\Gamma(\rho+\rho')}{\Gamma(\mu)}\int_0^1 {}_2F_1\left[\begin{matrix}\alpha,\beta; \\ \rho\;;\end{matrix}\; ux\right]\frac{d^{\mu-\rho}}{d(1-u)^{\mu-\rho}}$$

$$\times\left\{\frac{(1-u)^{\rho'-1}}{\Gamma(\rho')}\;{}_2F_1\left[\begin{matrix}\alpha',\beta'; \\ \rho'\;;\end{matrix}\; (1-u)y\right]\right\}du \quad (52)$$

Thus, by using the binomial theorem and (1), one obtain the required result (48).

Particular cases:

(I) If we put $\rho=\alpha$, $\rho'=\alpha'$ and $\mu=\beta$ in (48), which immediately reduced into (46).
(II) If we put $\rho=\beta$, $\rho'=\beta'$ and $\mu=\alpha$ in (48), which immediately reduced into (47).

In the present text, we derive the integral representations for the functions M_2 and M_{10} by using the technique of fractional integration by part as given below:

$$M_2\left[\begin{matrix}a,a',b,b',c,c'; \\ c+c'\;,\;\;d\;\;;\end{matrix}\; x,y\right] = \frac{\Gamma(c+c')}{\Gamma(c)\Gamma(c')}\int_0^1 u^{c'-1}(1-u)^{c-1}$$

$$\times M_1\left[\begin{matrix}a\;,a'\;,b\;,b'\;,c\;,c'\;; \\ d\;\;,\;\;c\;\;,\;\;c'\;\;;\end{matrix}\; ux,(1-u)y\right]du \quad (53)$$

where $Re(c')>0,\; Re(c)>0,\; |x|<1,\; |y|<1.$

Also,

$$M_{10}\left[\begin{matrix}a,b,c; \\ d,e,e';\end{matrix}\; x,y\right] = \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c')}\int_0^1 u^{c'-1}(1-u)^{d-c'-1}$$

$$\times\, M_{10}\left[\begin{matrix}a\;,b\;,c\;; \\ c'\;,e\;,e'\;;\end{matrix}\; ux,uy\right]du \quad (54)$$

where $Re(c')>0,\; Re(d-c')>0,\; |x|<1,\; |y|<1.$

Proof of (53): By using ([5], p. 73, Eq. 4.6)

$$M_2\left[\begin{matrix}a,a',b,b',c,c'; \\ c+c'\;\;,\;\;d\;;\end{matrix}\; x,y\right] = \frac{\Gamma(c)\Gamma(c')}{\Gamma(c+c')}\int_0^1 u^{c-1}(1-u)^{c'-1}$$

$$\times F_3\left[\begin{matrix}a\;,a'\;,b\;,b'\;; \\ d\;\;;\end{matrix}\; ux,(1-u)y\right]du \quad (55)$$

with the use of (1), we see that

$$(c')_r u^{c+r-1} = \frac{d^{c'-c}}{d(u)^{c'-c}}\left\{\frac{\Gamma(c+r)}{\Gamma(c')}u^{c'+r-1}\right\}$$

Substituting this in last integral (55) and using the formula (39), we get

$$M_2\left[\begin{matrix}a\;,a'\;,b\;,b'\;,c\;,c'\;; \\ c+c'\;\;,\;\;d\;\;;\end{matrix}\; x,y\right]$$

$$= \frac{\Gamma(c+c')}{\Gamma(c)\Gamma(c')}\sum_{r,s=0}^{\infty}\frac{(a)_r(a')_s(b)_r(b')_s(c)_r}{(d)_{r+s}(c')_r}$$

$$\times\int_0^1 u^{c'+r-1}\frac{d^{c'-c}}{d(1-u)^{c'-c}}\left\{(1-u)^{c'+s-1}\right\}du\frac{x^r}{r!}\frac{y^s}{s!} \quad (56)$$

Thus, with the aid of (1), we obtains the required result (53).

Proof of (54): By using ([5], p. 76, Eq. 4.14),

$$M_{10}\left[\begin{matrix}a,b,c; \\ d,e,e';\end{matrix}\; x,y\right] = \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)}\int_0^1 u^{c-1}(1-u)^{d-c-1}$$

$$\times F_4\left[\begin{matrix}a,b\;; \\ e,e';\end{matrix}\; ux,uy\right]du \quad (57)$$

Moreover by adopting the rule of (1), we see that

$$u^{c+r+s-1} = \frac{d^{c'-c}}{d(u)^{c'-c}} \left\{ \frac{\Gamma(c+r+s)}{\Gamma(c'+r+s)} u^{c'+r+s-1} \right\}$$

substituting this in last integral (57) and use the formula (39), we get

$$M_{10} \left[\begin{array}{ll} a\ , b\ , c\ ; & \\ & x, y \\ d\ , e\ , e'\ ; & \end{array} \right]$$

$$= \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}(b)_{r+s}(c)_{r+s}}{(c')_{r+s}(e)_r(e')_s} \int_0^1 u^{c'+r+s-1}$$

$$\times \frac{d^{c'-c}}{d(1-u)^{c'-c}} \left\{ (1-u)^{d-c-1} \right\} du \frac{x^r}{r!} \frac{y^s}{s!} \quad (58)$$

Further, again by using (1), we obtains the required result (54).

REFERENCES

[1] G. Lauricella, *Sulle funzioni ipergeometriche a piu variabili*, Rend. Circ. Mat. Palermo, 111-158, 1893.
[2] H. Exton, *Multiple Hypergeometric Functions and Applications,* Halsted Press (Ellis Harwood Ltd.) Chichester, 1976.
[3] H. M. Srivastava and P.W. Karlsson, *Multiple Gaussian hypergeometric series*, Halsted press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, 1985.
[4] H.M. Srivastava, *Generalized Neumann expansion involving hypergeometric functions*, Proc. Camb. Phil. Soc., 63, 425-429, 1967.
[5] M.A. Khan and G.S. Abukhammash, *On a generalization of Appell's functions of two variables,* Pro. Mathematica, Vol. XVI, Nos. 31-32, 61-83, 2002.
[6] P. Appell and J. Kampé de Fériet, *Fonctions hypérgeométriques et hyperspheriques,* Polynômes d' Hermite Gauthier-Villars, Paris, 1926.
[7] R.C. Pandey, *On certain hypergeometric transformations,* J. Math. Mech. 12, 113-118, 1963.
[8] S. Saran, *Hypergeometric functions of three variables,* Ganita, India, Vol.1, No.5, 83-90, 1954.
[9] S.F. Lacroix, *Traité du calculus differentiel calcul integral:* Mme, veconrcier, Tome Troisieme, seconde edition, 404-410, 1819.