Unsteady Poiseuille Flow of an Incompressible Elastico-Viscous Fluid in a Tube of Spherical Cross Section on a Porous Boundary

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***Abstract*—**Exact solution of an unsteady flow of elastico-viscous fluid through a porous media in a tube of spherical cross section under the influence of constant pressure gradient has been obtained in this paper. Initially, the flow is generated by a constant pressure gradient. After attaining the steady state, the pressure gradient is suddenly withdrawn and the resulting fluid motion in a tube of spherical cross section by taking into account of the porosity factor of the bounding surface is investigated. The problem is solved in two-stages the first stage is a steady motion in tube under the influence of a constant pressure gradient, the second stage concern with an unsteady motion. The problem is solved employing separation of variables technique. The results are expressed in terms of a non-dimensional porosity parameter (K) and elastico-viscosity parameter (β), which depends on the Non-Newtonian coefficient. The flow parameters are found to be identical with that of Newtonian case as elastic-viscosity parameter tends to zero and porosity tends to infinity. It is seen that the effect of elastico-viscosity parameter, porosity parameter of the bounding surface has significant effect on the velocity parameter.

***Keywords*—**Elastico-viscous fluid, Porous media, Second order fluids, Spherical cross-section.

I. Introduction

FLOW through porous media has been the subject of considerable research activity in recent years because of its several important applications notably in the flow of oil through porous rock, the extraction of geothermal energy from the deep interior of the earth to the shallow layers, the evaluation of the capability of heat removal from particulate nuclear fuel debris that may result from a hypothetical accident in a nuclear reactor, the filtration of solids from liquids, flow of liquids through ion exchange beds, drug permeation through human skin, chemical reactor for economical separation or purification of mixtures and so on.

In many chemical processing industries, slurry adheres to the reactor vessels and gets consolidated. As a result of this, the chemical compounds within the reactor vessel percolates through the boundaries causing loss of production and then consuming more reaction time. In view of such technological and industrial importance wherein the heat and mass transfer takes place in the chemical industry, the problem by considering the permeability of the bounding surfaces in the reactors attracted the attention of several investigators. An important application is in the petroleum industry, where crude oil is tapped from natural underground reservoirs in which oil is entrapped. Since the flow behaviour of fluids in petroleum reservoir rock depends, to a large extent, on the properties of the rock, techniques that yield new or additional information on the characteristics of the rock would enhance the performance of the petroleum reservoirs. A related bio-mechanical application is the flow of fluids in the lungs, blood vessels, arteries and so on, where the fluid is bounded by two layers which are held together by a set of fairly regularly spaced tissues.

Viscous fluid flow over wavy wall had attracted the attention of relatively few researchers although the analysis of such flows finds application in different areas, such as transpiration cooling of re-entry vehicles and rocket boosters, cross hatching on ablative surfaces and film vaporization in combustion chambers etc. Especially, where the heat and mass transfer takes place in the chemical processing industry, the problem by considering the permeability of the bounding surface in the reactors assumes greater significance. Many materials such as drilling muds, clay coatings and other suspensions, certain oils and greases, polymer melts, elastomers and many emulsions have been treated as non-Newtonian fluids. Because of the difficulty to suggest a single model, which exhibits all properties of non-Newtonian fluids, they cannot be described simply as Newtonian fluids and there has been much confusion over the classification of non-Newtonian fluids. However, non-Newtonian fluids may be classified as (i) fluids for which the shear stress depends only on the rate of shear; (ii) fluids for which the relation between shear stress and shear rate depends on time; (iii) the visco-elastic fluids, which possess both elastic and viscous properties.

Because of the great diversity in the physical structure of non-Newtonian fluids, it is not possible to recommend a single constitutive equation as the equation for use in the cases described in (i) – (iii). For this reason, many non-Newtonian models or constitutive equations have been proposed and most of them are empirical or semi-empirical. For more general three dimensional representation, the method of continuum mechanics is needed [1]. Although many constitutive equations have been suggested, many questions are still unsolved. Some of the continuum models do not give satisfactory results in accordance with available experimental data. For this reason, in many practical applications, empirical or semi-empirical equations have been used.

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It has been shown that for many types of problems in which the flow is slow enough in the visco-elastic sense, the results given by Oldroyd's constitutive equations will be substantially equal to those of the second or third order Rivilin – Ericksen constitutive equations [2]. Thus if this is the sense in which the solutions to which problems are to be interpreted, it would seem reasonable to use the second or third order constitutive equations in carrying out the calculations. This is particularly so in view of the fact that, the calculation will generally be still simpler. For this reason, in this paper, the second order fluid model is used. The constitutive equation for the fluids of second grade (or second order fluids) is a linear relationship between the stress, the first Rivlin - Ericksen tensor, its square and the second Rivlin – Ericksen tensor. The constitutive equation has three coefficients. There are some restrictions on these coefficients due to the Clausius – Duhem inequality and the assumption that the Helmholtz free energy is a minimum in equilibrium. A comprehensive discussion on the restrictions for these coefficients has been given in [3], [4]. One of these coefficients represents the viscosity coefficient in a way similar to that of a Newtonian fluid in the absence of the other two coefficients. The restrictions on these two coefficients have not been confirmed by experiments and the sign of these material moduli is the subject of much controversy [5]. The equation of the motion of incompressible second grade fluids, in general, is of higher order than the Navier – Stokes equation. The Navier - Stokes equation is second order partial differential equation, but the equation of motion of a second order fluid is a third order partial differential equation. A marked difference between the case of the Navier – Stokes theory and that for fluids of second grade is that ignoring the nonlinearity in the Navier – Stokes equation does not lower the order of the equation however, ignoring the higher order nonlinearities in the case of the second grade fluid, reduces the order of the equation. Exact solutions are very important for many reasons. They provide a standard for checking the accuracies of many approximate methods such as numerical and empirical. Although computer techniques make the complete numerical integration of the non-linear equations feasible, the accuracy of the results can be established by a comparison with an exact solution. Many attempts to collect the exact solution of the nonlinear equations for unsteady flow of second grade fluid have been by different researcher for different geometries.

In view of several industrial and technological importance, [6] studied the problem of the exact solutions of two dimensional flows of a second order incompressible fluid by considering the rigid boundaries. Later, a linear analysis of the compressible boundary layer flow over a wall was presented by [7]. Subsequently, [8] studied the problem of Rayleigh for wavy wall while [9] examined the effect of small amplitude wall waviness upon the stability of the laminar boundary layer. Further, the problem of free convective heat transfer in a viscous incompressible fluid confined between vertical wavy wall and a particle flat wall was examined by [10], [11]. Later, [12] studied the free convective flow of a viscous incompressible fluid in porous medium between two long vertical wavy walls. Subsequently, [13] had examined the problem of MHD flow with slip effects and temperature dependent heat source in a viscous incompressible fluid confined between a long vertical wall and a parallel flat plate. Later, [14] examined the problem of elastico-viscous fluid of second order type where the bounding surface is porous and subjected to sinusoidal disturbances. Subsequently, [15] studied the unsteady poiseuille flow of second order fluid in a tube of elliptical cross section on the porous boundary. Recently, [23] had examined the problem of unsteady flow of an incompressible viscous electrically conducting fluid in the tube of elliptical cross section under the influence of the magnetic field.

In all above investigations, the fluid under consideration was viscous incompressible fluid and one of the bounding surfaces has a wavy character or bounding surface subjected to sinusoidal disturbances. In all of the above situations, not much of attention has been paid on the study of unsteady flow of second order fluid in an infinitely long tube of circular, elliptical or spherical cross section on the porous boundary. Therefore, an attempt has been made to study the effects of the flow of incompressible elastico-viscous fluid of second order type in an infinitely long tube of spherical cross section is considered under constant pressure gradient on the porous boundary. The results are expressed in terms of a non-dimensional porosity parameter, which depends on the non-Newtonian coefficient. It is noticed that the flow properties are identical with those in the Newtonian case ($\beta = 0, K \to \infty$).

II. Mathematical Formulation of the Problem

In the sense of [16] a simple material is a substance for which stress can be determined with entire knowledge of the history of the strain. This is called simple fluid, if it has property that at all local states, with the same mass density, are intrinsically equal in response, with all observable differences in response being due to definite differences in the history. For any given history $g(s)$, a retarded history $g_\alpha(s)$ can be defined as:

$$g_\alpha(s) = g(\alpha s);\ 0 < s < \infty, 0 < \alpha \le 1 \tag{1}$$

α being termed as a retardation factor. Assuming that the stress is more sensitive to recent deformation that to the deformations at distant past, it has been established by [17]that the theory of simple fluids yields the theory of perfect fluids as $\alpha \to 0$ and that of Newtonian Fluids as a correction (up to the order of α) to the theory of the perfect fluids. Neglecting all the terms of the order of higher than two in α, We have incompressible elastic-viscous fluid of second order type whose constitutive relation is governed by:

$$S = -PI + \phi_1 E_{ij}^{(1)} + \phi_2 E_{ij}^{(2)} + \phi_3 E_{ij}^{(1)^2} \tag{2}$$

where

$$E_{ij}^{(1)} = U_{i,j} + U_{j,i} \quad (3)$$

and

$$E_{ij}^{(2)} = A_{i,j} + A_{j,i} + 2U_{m,i}U_{m,j} \quad (4)$$

In the above equations, S is the stress-tensor, U_i and A_i are the components of velocity and acceleration in the direction of the i^{th} coordinate X_i while P is indeterminate hydrostatic pressure. The coefficients ϕ_1, ϕ_2 and ϕ_3 are material constants. The constitutive relation for general [18] fluid also reduces to (2), when the squares and higher orders of $E^{(2)}$ are neglected, while the coefficients being constants. Also the non-Newtonian models considered by [19] could be obtained from (2), when $\phi_2 = 0$ and naming ϕ_3 as the coefficient of cross viscosity. With reference to the [18] fluids, ϕ_2 be called as the coefficient of elastico- viscosity.

The Clausius-Duhem inequality and the assumption that the Helmholtz free energy is minimum in equilibrium provide the following restriction [3].

$$\phi_1 \geq 0,\ \phi_2 \geq 0,\ \phi_1 + \phi_2 = 0$$

The condition $\phi_1 + \phi_2 = 0$ is consequence of the Clausius-Duhem inequality and the condition $\phi_2 \geq 0$ follows the requirement that the Helmholtz free energy is a minimum in equilibrium. A comprehensive discussion on the restrictions for ϕ_1, ϕ_2 and ϕ_3 can be found in the work by [4]. The sign of the material moduli ϕ_1, ϕ_2 is the subject of much controversy [5]. In the experiments on several non-Newtonian fluids, the experimentalists have not confirmed these restrictions ϕ_1 and ϕ_2.

If $\mathbf{V}(U_1, U_2, U_3)$ is the velocity component and **F** (F_x, F_y, F_z) are the body forces acting on the system, then the equation of motion in X, Y and Z directions is given by:

$$\rho \frac{DU_1}{DT} = \rho F_X + \frac{\partial S_{XX}}{\partial X} + \frac{\partial S_{XY}}{\partial Y} + \frac{\partial S_{XZ}}{\partial Z} \quad (5)$$

$$\rho \frac{DU_2}{DT} = \rho F_Y + \frac{\partial S_{YX}}{\partial X} + \frac{\partial S_{YY}}{\partial Y} + \frac{\partial S_{YZ}}{\partial Z} \quad (6)$$

$$\rho \frac{DU_3}{DT} = \rho F_Z + \frac{\partial S_{ZX}}{\partial X} + \frac{\partial S_{ZY}}{\partial Y} + \frac{\partial S_{ZZ}}{\partial Z} \quad (7)$$

where

$$\frac{D}{DT} = \frac{\partial \mathbf{V}}{\partial T} + \mathbf{V} \cdot \nabla \mathbf{V}$$

If the bounding surface is porous, then the rate of percolation of the fluid is directly proportional to the cross sectional area of the filter bed and the total force, say the sum of the pressure gradient and the gravity force [20].

$$q = CA\left(\frac{P_1 - P_2}{H_1 - H_2} + \rho G\right) \quad (8)$$

where A is the cross sectional area of the filter bed, $C = \frac{k}{\mu}$ in which k is the permeability of the material and μ is the coefficient of viscosity and q is the flux of the fluid. A straight forward generalization of (8) yields

$$\mathbf{V} = -\frac{k}{\mu}[\nabla P + \rho G \boldsymbol{\eta}] \quad (9)$$

where $\mathbf{V}$ is the velocity vector and $\boldsymbol{\eta}$ is the unit vector along the gravitational force. If any other external forces are acting on the system, instead of gravitational force, then we have

$$\mathbf{V} = -\frac{k}{\mu}[\nabla P - \rho \mathbf{F}] \quad (10)$$

In the absence of external forces, $\mathbf{V} = -\frac{k}{\mu}\nabla P$ this gives

$$\nabla P = -\frac{\mu}{k}\mathbf{V}.$$

Therefore, the net resulting equations (in the dimensional form) of motions in the X, Y and Z directions and when the bounding surface is porous are given by

$$\rho \frac{DU_1}{DT} = \rho F_X + \frac{\partial S_{XX}}{\partial X} + \frac{\partial S_{XY}}{\partial Y} + \frac{\partial S_{XZ}}{\partial Z} - \frac{\mu}{k}U_1 \quad (11)$$

$$\rho \frac{DU_2}{DT} = \rho F_Y + \frac{\partial S_{YX}}{\partial X} + \frac{\partial S_{YY}}{\partial Y} + \frac{\partial S_{YZ}}{\partial Z} - \frac{\mu}{k}U_2 \quad (12)$$

$$\rho \frac{DU_3}{DT} = \rho F_Z + \frac{\partial S_{ZX}}{\partial X} + \frac{\partial S_{ZY}}{\partial Y} + \frac{\partial S_{ZZ}}{\partial Z} - \frac{\mu}{k}U_3 \quad (13)$$

Introducing the following non dimensional variables as:

$$U_i = \frac{\phi_1 u_i}{\rho L} \quad T = \frac{\rho L^2 t}{\phi_1} \quad \phi_2 = \rho L^2 \beta \quad P = \frac{\phi_1^2 p}{\rho L^2},\ \frac{X_i}{L} = x_i,\ \frac{Y_i}{L} = y_i,$$

$$\phi_3 = \rho L^2 \upsilon_c \quad A_i = \frac{\phi_1^2 a_i}{\rho^2 L^3} \quad S_{i,j} = \frac{\phi_1^2 s_{i,j}}{\rho L^2} \quad E_{i,j}^{(1)} = \frac{\phi_1 e_{i,j}^{(1)}}{\rho L^2} \quad M = \frac{\phi_1 m}{L^2}$$

$$E_{i,j}^{(2)} = \frac{\phi_1^2 e_{i,j}^{(2)}}{\rho^2 L^4} \quad k = \frac{L^2 K}{\phi_1} \quad F_i = \frac{\phi_1^2}{\rho L^3} f_i \quad \frac{Z_i}{L} = z_i$$

where T is the (dimensional) time variable, and ρ the mass density and L a characteristic length.

We consider a class of plane flows given by the velocity components in the directions of rectangular Cartesian coordinates x and y.

$$u_1 = u(y, z, r, t) \text{ and } u_2 = 0 \quad (14)$$

The velocity field given by (14) identically satisfies the incompressibility condition. The stress can now be obtained in the non-dimensional form as:

$$s_{xx} = -p + \upsilon_c (\frac{\partial u}{\partial y})^2 \quad (15)$$

$$s_{yy} = -p + (\upsilon_c + 2\beta)(\frac{\partial u}{\partial y})^2 \quad (16)$$

$$s_{xy} = \frac{\partial u}{\partial y} + \beta \frac{\partial}{\partial y}(\frac{\partial u}{\partial t}) \quad (17)$$

In view of the above, the equations of motion in the present case of porous boundary will yield

X - Component:

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial u^2}{\partial r^2} + \beta \frac{\partial}{\partial t}(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial r^2}) - (\frac{1}{K})u \quad (18)$$

Y - Component

$$0 = -\frac{\partial p}{\partial y} + (2\beta + \upsilon_c)\frac{\partial}{\partial y}(\frac{\partial u}{\partial y})^2 + \frac{\partial}{\partial z}(\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}\frac{\partial u}{\partial r}) \quad (19)$$

Z – Component

$$0 = -\frac{\partial p}{\partial z} + (2\beta + \upsilon_c)\frac{\partial}{\partial y}(\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}\frac{\partial u}{\partial r}) + \frac{\partial}{\partial z}(\frac{\partial u}{\partial z})^2 \quad (20)$$

Equation (18) shows that $-\frac{\partial p}{\partial x}$ must be independent of space variables and hence may be taken as $\xi(t)$; (19) now yields

$$p = p_0(t) - \xi(t)x + (\upsilon_c + 2\beta)(\frac{\partial u}{\partial y})^2$$

$$\frac{\partial p}{\partial y} = 0 \text{ and } \frac{\partial p}{\partial z} = 0 \cdot$$

showing that $p = p(x)$. Therefore (18)-(20) reduce to single equation the flow characterized by the velocity is given by:

$$\frac{\partial u}{\partial t} = -\frac{dp}{dx} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial r^2} + \beta \frac{\partial}{\partial t}(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial r^2}) - (\frac{1}{K})u \quad (21)$$

where K is the non-dimensional porosity constant. It may be noted that the presence of β changes the order of differential from two to three.

Consider the flow of an incompressible unsteady flow of an elastico-viscous fluid, isothermal second order fluid in an infinitely long tube, under constant pressure gradient and negligible gravity. The tube has a spherical cross-section. The flow is considered to be unsteady and two dimensional. Accordingly the flow velocity u has one non-vanishing component u_x, which depends on the coordinates y, z and r given in (21). Boundary conditions require that the flow velocity vanishes at the wall of the tube, i.e. on the sphere $y^2 + z^2 + r^2 = 1$ and that the gradient of the velocity vanishes at the center of the tube, $y = z = r = 0$.

III. Solution of the Problem

Erdogan has presented the unsteady flows of an incompressible viscous fluid in rectangular and circular cross-sections. In this paper we have solved unsteady two dimensional flow problem exactly using separation of variables [21]. To reduce the unsteady problem given in (21) into steady and transient problems using following transformation

$$u(y,z,r,t) = f(y,z,r) + g(y,z,r,t) \quad (22)$$

Using (22) in (21) we get,

$$\frac{\partial (f+g)}{\partial t} = -\frac{dp}{dx} + \frac{\partial^2 (f+g)}{\partial y^2} + \frac{\partial^2 (f+g)}{\partial z^2} + \frac{\partial^2 (f+g)}{\partial r^2}$$
$$\beta \frac{\partial}{\partial t}(\frac{\partial^2 (f+g)}{\partial y^2} + \frac{\partial^2 (f+g)}{\partial z^2} + \frac{\partial^2 (f+g)}{\partial r^2}) - (\frac{1}{K})(f+g)$$

On simplification we get following equation.

$$\frac{\partial g}{\partial t} = -\frac{dp}{dx} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 g}{\partial z^2} + \frac{\partial^2 g}{\partial r^2} + \frac{\partial^2 f}{\partial r^2}$$
$$\beta \frac{\partial}{\partial t}(\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 g}{\partial z^2} + \frac{\partial^2 g}{\partial r^2} + \frac{\partial^2 f}{\partial r^2}) - (\frac{1}{K})f - (\frac{1}{K})g \quad (23)$$

Comparing the terms from the above (23) we get steady and unsteady problems. The steady state problem is related to the function $f(y,z,r)$ in (23) is given by:

$$-\frac{dp}{dx} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 f}{\partial r^2} - (\frac{1}{K})f = 0$$

After rearranging terms in above steady problem we get following equation:

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 f}{\partial r^2} = \frac{dp}{dx} + (\frac{1}{K})f \quad (24)$$

To solve the steady state problem by assuming solution $f(y,z)$ of the following form

$$f(y,z,r) = p(1 - y^2 - z^2 - r^2) \quad (25)$$

Using (25) in (24) we get steady state solution given by

$$\frac{\partial^2}{\partial y^2}[p(1-y^2-z^2-r^2)] + \frac{\partial^2}{\partial z^2}[p(1-y^2-z^2-r^2)] = \frac{dp}{dx} + \frac{1}{K}p(1-y^2-z^2-r^2)$$

$$-2p - 2p - 2p = \frac{dp}{dx} + \frac{1}{K}p(1-y^2-z^2-r^2)$$

$$-6p - \frac{1}{K}p(1-y^2-z^2-r^2) = \frac{dp}{dx}$$

On simplification of above equation we get the value of p

is given by

$$p = -\frac{\frac{dp}{dx}}{\left[6 + \frac{1}{K}(1 - y^2 - z^2 - r^2)\right]}$$

Now putting the value of p in (25) we get steady state solution.

$$f(y,z,r) = -\frac{\frac{dp}{dx}(1 - y^2 - z^2 - r^2)}{\left[6 + \frac{1}{K}(1 - y^2 - z^2 - r^2)\right]} \quad (26)$$

The unsteady state problem is related to the function $g(y,z,t)$ in (23) is given by:

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} + \frac{\partial^2 g}{\partial r^2} + \beta \frac{\partial}{\partial t}(\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} + \frac{\partial^2 g}{\partial r^2}) - \frac{1}{K} g \quad (27)$$

Subject to following boundary and initial conditions

$$g(1,z,r,t) = 0,\ g(y,1,r,t) = 0,\ g(y,z,1,t) = 0$$
$$g(y,z,r,0) = g(y,z,r)\ \frac{\partial g}{\partial y}(0,z,r,t) = 0,\ \frac{\partial g}{\partial z}(y,0,r,t) = 0$$
$$\frac{\partial g}{\partial r}(y,z,0,t) = 0 \quad (28)$$

To solve above unsteady state IBVP using separation of variables method and assuming solution $g(y,z,r,t)$ of the following form.

$$g(y,z,r,t) = Y(y)Z(z)R(r)T(t) \quad (29)$$

Using (29) in (27) we get

$$\frac{\partial}{\partial t}(YZRT) = \frac{\partial^2}{\partial y^2}(YZRT) + \frac{\partial^2}{\partial z^2}(YZRT) + \frac{\partial^2}{\partial r^2}(YZRT) + \beta \frac{\partial}{\partial t}(\frac{\partial^2}{\partial y^2}(YZRT)$$
$$+ \frac{\partial^2}{\partial z^2}(YZRT) + \frac{\partial^2}{\partial r^2}(YZRT)) - \frac{1}{K} YZRT$$

On simplification by using partial differentiation we get

$$YZR\frac{\partial T}{\partial t} = ZRT\frac{\partial^2 Y}{\partial y^2} + YRT\frac{\partial^2 Z}{\partial z^2} + YZT\frac{\partial^2 R}{\partial r^2} +$$
$$\beta\frac{\partial}{\partial t}(ZRT\frac{\partial^2 Y}{\partial y^2} + YRT\frac{\partial^2 Z}{\partial z^2} + YZT\frac{\partial^2 R}{\partial r^2}) - \frac{1}{K}YZRT$$
$$YRZ\frac{\partial T}{\partial t} = ZRT\frac{\partial^2 Y}{\partial y^2} + YRT\frac{\partial^2 Z}{\partial z^2} + YZT\frac{\partial^2 R}{\partial r^2} +$$
$$\beta\frac{\partial T}{\partial t}(ZRT\frac{\partial^2 Y}{\partial y^2} + YRT\frac{\partial^2 Z}{\partial z^2} + YZT\frac{\partial^2 R}{\partial r^2}) - \frac{1}{K}YZRT$$
$$YZR\frac{\partial T}{\partial t} = (T + \beta\frac{\partial T}{\partial t})(ZR\frac{\partial^2 Y}{\partial y^2} + YR\frac{\partial^2 Z}{\partial z^2} + YZ\frac{\partial^2 R}{\partial r^2}) - \frac{1}{K}YZRT$$

Dividing by YZR and rearranging the terms

$$\frac{\partial T}{\partial t} = (T + \beta\frac{\partial T}{\partial t})(\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + \frac{1}{R}\frac{\partial^2 R}{\partial r^2}) - \frac{1}{K}T$$
$$\frac{\frac{\partial T}{\partial t} + \frac{1}{K}T}{T + \beta\frac{\partial T}{\partial t}} = (\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + \frac{1}{R}\frac{\partial^2 R}{\partial r^2}) = -P^2$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + \frac{1}{R}\frac{\partial^2 R}{\partial r^2} = -P^2 \text{ and } \frac{\frac{\partial T}{\partial t} + \frac{1}{K}T}{T + \beta\frac{\partial T}{\partial t}} = -P^2$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -J^2,\ \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -L^2,\ \frac{1}{R}\frac{\partial^2 R}{\partial r^2} = -M^2$$

$$(1 + \beta P^2)\frac{\partial T}{\partial t} + (\frac{1}{K} + P^2)T = 0$$

The modified system of differential initial and boundary conditions are given by

$$Y'' + J^2 Y = 0,\ Y'(0) = Y(1) = 0,$$
$$Z'' + L^2 Z = 0,\ Z'(0) = Z(1) = 0,$$
$$R'' + M^2 R = 0,\ R'(0) = R(1) = 0, \quad (30)$$
$$T' + \left[\frac{(\frac{1}{K} + P^2)}{(1 + \beta P^2)}\right]T = 0,\quad T(0) = -f(y,z,r).$$

The solutions obtained for differential equations are

$$Y_m = B_m \cos\frac{(2m+1)\pi y}{2},\quad m = 0,1,2,........$$
$$Z_n = D_n \cos\frac{(2n+1)\pi z}{2},\ n = 0,1,2,........$$
$$R_o = E_o \cos\frac{(2n+1)\pi z}{2},\ o = 0,1,2,........ \quad (31)$$
$$T_{mno} = \exp\left[-\frac{(\frac{1}{K} + (\frac{(2m+1)\pi}{2})^2 + (\frac{(2n+1)\pi}{2})^2 + (\frac{(2o+1)\pi}{2})^2)}{(1 + \beta((\frac{(2m+1)\pi}{2})^2 + (\frac{(2n+1)\pi}{2})^2 + (\frac{(2o+1)\pi}{2})^2))}t\right]$$

The solution of the unsteady problem is given by

$$g(y,z,r,t) = \sum_{m=0}^{\infty}\ \sum_{n=0}^{\infty}\sum_{o=0}^{\infty} D_{mno}\cos\frac{(2m+1)\pi y}{2}$$
$$\cos\frac{(2n+1)\pi z}{2}\cos\frac{(2o+1)\pi r}{2}$$
$$\exp\left[-\frac{(\frac{1}{K} + (\frac{(2m+1)\pi}{2})^2 + (\frac{(2n+1)\pi}{2})^2 + (\frac{(2o+1)\pi}{2})^2)}{(1 + \beta((\frac{(2m+1)\pi}{2})^2 + (\frac{(2n+1)\pi}{2})^2 + (\frac{(2o+1)\pi}{2})^2))}t\right]$$

where

$$D_{mno} = \frac{8}{ab}\int_0^1\int_0^1\int_0^1 -f(y,z,r)\cos\frac{(2m+1)\pi y}{2}\cos\frac{(2n+1)\pi z}{2}\cos\frac{(2o+1)\pi r}{2}\,dydzdr \qquad (32)$$

and the complete velocity distribution is given by

$$u(y,z,r,t) = f(y,z,r) + g(y,z,r,t)$$

$$u(y,z,r,t) = -\frac{\frac{dp}{dx}(1-y^2-z^2-r^2)}{\left[6+\frac{1}{K}(1-y^2-z^2-r^2)\right]} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{o=0}^{\infty} D_{mno}\cos\frac{(2m+1)\pi y}{2}\cos\frac{(2n+1)\pi z}{2}\cos\frac{(2o+1)\pi r}{2}$$
$$\exp\left[-\frac{\left(\frac{1}{K}+\left(\frac{(2m+1)\pi}{2}\right)^2+\left(\frac{(2n+1)\pi}{2}\right)^2+\left(\frac{(2o+1)\pi}{2}\right)^2\right)}{\left(1+\beta\left(\left(\frac{(2m+1)\pi}{2}\right)^2+\left(\frac{(2n+1)\pi}{2}\right)^2+\left(\frac{(2o+1)\pi}{2}\right)^2\right)\right)}t\right]$$

VI. Conclusions

In this paper, a problem is studied in order to show the effect of the applied pressure gradient in a channel of spherical cross-section on unsteady flow of a fluid of second order with bounding surface is porous. When $K \to \infty$ the results obtained for the velocity field in agreement to that of [22]. The case of Newtonian fluid can be realized as $\beta \to 0$ and $K \to \infty$.

Appendix

ϕ_1	Coefficient of viscosity
$g_\alpha(s)$	Retarded history
A_i	Acceleration component in the i[th] coordinate
L	Characteristic Length
ϕ_3	Coefficient of cross-viscosity
ϕ_2	Coefficient of elastico-viscosity
ρ	Density of the fluid
a_i	Dimensionless acceleration component in the i[th] direction
υ_c	Dimensionless cross viscosity parameter
β	Dimensionless elastico-viscosity parameter
F	Dimensionless External force applied
p	Dimensionless indeterminate hydrostatic pressure
K	Dimensionless porosity factor
u_i	Dimensionless velocity component along the i[th] coordinate
$g(s)$	Given history
P	Indeterminate hydrostatic pressure
α	Retardation factor
S	Stress tensor
T	Time parameter
U_i	Velocity component in the i[th] direction

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