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Solving Linear Matrix Equations by Matrix Decompositions

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Abstract—In this paper, a system of linear matrix equations is considered. A new necessary and sufficient condition for the consistency of the equations is derived by means of the generalized singular-value decomposition, and the explicit representation of the general solution is provided.

Keywords—Matrix equation, Generalized inverse, Generalized singular-value decomposition.

I. INTRODUCTION

We consider the solution of the linear matrix equations

$$\begin{cases} A_1X = C_1, \\ A_2Y = C_2,\ YB_2 = D_2,\ Y = Y^{\mathrm{H}}, \\ A_3Z = C_3,\ ZB_3 = D_3,\ Z = Z^{\mathrm{H}}, \\ A_4X + (A_4X)^{\mathrm{H}} + B_4YB_4^{\mathrm{H}} + C_4ZC_4^{\mathrm{H}} = D_4, \end{cases} \tag{1}$$

where

$$\begin{aligned} &A_1 \in \mathbf{C}^{a_1\times m},\ C_1 \in \mathbf{C}^{a_1\times n}, \\ &A_2,\ C_2 \in \mathbf{C}^{a_2\times p},\ B_2,\ D_2 \in \mathbf{C}^{p\times b_2}, \\ &A_3,\ C_3 \in \mathbf{C}^{a_3\times q},\ B_3,\ D_3 \in \mathbf{C}^{q\times b_3}, \\ &A_4 \in \mathbf{C}^{n\times m},\ B_4 \in \mathbf{C}^{n\times p},\ C_4 \in \mathbf{C}^{n\times q}, \end{aligned}$$

and

$$D_4 = D_4^{\mathrm{H}} \in \mathbf{C}^{n\times n}.$$

Solvability and solutions of matrix equations have been one of principle topics in matrix analysis and its applications. For instance, Mitra [1] considered solutions with fixed ranks for the matrix equations $AX = B$ and $AXB = C$, Mitra [2] gave common solutions of minimal rank of the pair of matrix equations $AX = C, XB = D$. Uhlig [3] gave the maximal and minimal ranks of solutions of the equation $AX = B$. Mitra [4] examined common solutions of minimal rank of the pair of matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$. In 2006, Lin and Wang in [5] studied the extreme ranks of solutions to the system of matrix equations $A_1X = C_1, XB_2 = C_2, A_3XB_3 = C_3$ over an arbitrary division ring, which was investigated in [6] and [7]. Liu [8] derived the maximal and minimal ranks of least squares solutions for $AXB = C$ using the matrix rank method and the normal equation. Cvetković-Ilić [9], Peng, Hu and Zhang [10] considered the reflexive and anti-reflexive solutions of the matrix equation $AXB = C$ by means of generalized inner inverse and the generalized singular-value decomposition. In the papers [11–13], necessary and sufficient conditions for the existence of symmetric and anti-symmetric solutions of the equation $AXB = C$ were investigated.

Wu [14] studied Re-pd solutions of the equation $AX = C$ and Wu and Cain [15] found the set of all complex Re-nnd matrices X such that $XB = C$ and presented a criterion for Re-nndness. Größ [16] gave an alternative approach, which simultaneously delivers explicit Re-nnd solutions and gave a corrected version of some results from [15]. Recently, in [17] and [18], the common Re-nnd and Re-pd solutions of the matrix equations $AX = C, XB = D$, where $A, C \in \mathbf{C}^{n\times m}$ and $B, D \in \mathbf{C}^{m\times n}$, are considered by virtue of the maximal and minimal ranks of matrix polynomials. Wang and Yang [19] presented criteria for 2×2 and 3×3 partitioned matrices to be Re-nnd, found necessary and sufficient conditions for the existence of Re-nnd solutions of $AXB = C$ and derived an expression for these solutions. In the special case that A and B are both nonnegative matrices, Cvetković-Ilić [20] put forward a necessary and sufficient condition for the existence of Re-nnd solutions of $AXB = C$ in terms of g-inverses. Zhang, Sheng and Xu [21] generalized the main results of [20] from the finite-dimensional case to the Hilbert space operator case.

Matrix equations such as $A_4X+(A_4X)^{\mathrm{H}} = D_4$, $B_4YB_4^{\mathrm{H}} = D_4$ and $B_4YB_4^{\mathrm{H}}+C_4ZC_4^{\mathrm{H}} = D_4$, which are the special cases of (1), arise in a number of practical applications in linear system theory, numerical analysis and structural dynamics, and have been studied by Braden [22], Djordjević [23], Yuan [24, 25], Dai and Lancaster [26], Baksalary [27], Größ [28], Liu, Tian and Takane [29], Liao and Bai [30], Deng and Hu [31] and Liu and Tian [32], and so forth.

Very recently, Wang and He [33] derived the solvability conditions and the expression of the general solution to the matrix equations of (1) by virtue of the maximal and minimal ranks of matrix polynomials. In this paper, we will provide a new approach based on the generalized inverses and the generalized singular-value decomposition (GSVD) to solve (1). In Section II, we establish a necessary and sufficient

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condition for the existence of the solution of (1) directly by means of the GSVD, and construct the explicit representation of the general solution when it is solvable. Throughout this paper, we denote the complex $m \times n$ matrix space by $\mathbf{C}^{m\times n}$, the conjugate transpose and the Moore-Penrose generalized inverse of a complex matrix A by A^{H} and A^{+}, respectively. I_n represents the identity matrix of size n. Furthermore, for a matrix $A \in \mathbf{C}^{m\times n}$, let E_A and F_A stand for the two orthogonal projectors: $E_A = I_m - AA^{+}$ and $F_A = I_n - A^{+}A$.

II. The Solution of (1)

Let $G_1 \in \mathbf{C}^{n\times q}, G_2 \in \mathbf{C}^{n\times d}$, then the GSVD (see, e.g., [34–36]) of the matrix pair (G_1, G_2) is of the form

$$G_1 = N\Omega_1 P^{\mathrm{H}},\ G_2 = N\Omega_2 Q^{\mathrm{H}}, \tag{2}$$

where $P \in \mathbf{C}^{q\times q}, Q \in \mathbf{C}^{d\times d}$ are unitary matrices and $N \in \mathbf{C}^{n\times n}$ is a nonsingular matrix, and

$$\Omega_1 = \begin{array}{cc} \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & S_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] & \begin{array}{c} h-t \\ t \\ e-h \\ n-e \end{array} \\ \begin{array}{ccc} h-t & t & q-h \end{array} & \end{array},$$

$$\Omega_2 = \begin{array}{cc} \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{array}\right] & \begin{array}{c} h-t \\ t \\ e-h \\ n-e \end{array} \\ \begin{array}{ccc} u & t & e-h \end{array} & \end{array},$$

$$u = d + h - e - t,$$
$$h = \mathrm{rank}(G_1),\ e = \mathrm{rank}[G_1, G_2],$$

and

$$S_1 = \mathrm{diag}\{\gamma_1, \cdots, \gamma_t\},$$
$$S_2 = \mathrm{diag}\{\delta_1, \cdots, \delta_t\}$$

with

$$1 > \gamma_1 \geq \cdots \geq \gamma_t > 0,$$
$$0 < \delta_1 \leq \cdots \leq \delta_t < 1,$$
$$\gamma_i^2 + \delta_i^2 = 1\ (i = 1, \cdots, t).$$

Lemma 1: [37] If $A \in \mathbf{C}^{m\times n}, H \in \mathbf{C}^{m\times l}$, then

$$AZ = H$$

has a solution $Z \in \mathbf{C}^{n\times l}$ if and only if

$$AA^{+}H = H.$$

In this case, the general solution of the equation can be described as

$$Z = A^{+}H + F_A L,$$

where $L \in \mathbf{C}^{n\times l}$ is an arbitrary matrix.

Lemma 2: [38] Let $A, B \in \mathbf{C}^{n\times p}$, then the matrix equation

$$AY = B$$

has a Hermitian solution $Y \in \mathbf{C}^{p\times p}$ if and only if

$$BA^{\mathrm{H}} = AB^{\mathrm{H}},\ AA^{+}B = B,$$

in which case, the general Hermitian solution is

$$Y = A^{+}B + F_A(A^{+}B)^{\mathrm{H}} + F_A K F_A,$$

where $K \in \mathbf{C}^{p\times p}$ is an arbitrary Hermitian matrix.

Lemma 3: [22, 24] Let $A \in \mathbf{C}^{m\times n}, D \in \mathbf{C}^{m\times m}$. Then the matrix equation

$$AX^{\mathrm{H}} + XA^{\mathrm{H}} = D$$

has a solution X if and only if

$$D = D^{\mathrm{H}},\ E_A D E_A = 0,$$

in which case, the general solution is

$$X = \tfrac{1}{2}D(A^{+})^{\mathrm{H}} + \tfrac{1}{2}E_A D(A^{+})^{\mathrm{H}} + V - \tfrac{1}{2}VA^{+}A$$
$$-\tfrac{1}{2}AV^{\mathrm{H}}(A^{+})^{\mathrm{H}} - \tfrac{1}{2}E_A V A^{+}A,$$

where $V \in \mathbf{C}^{m\times n}$ is an arbitrary matrix.

It follows from Lemma 1 that the matrix equation $A_1 X = C_1$ has a solution $X \in \mathbf{C}^{m\times n}$ if and only if

$$A_1 A_1^{+} C_1 = C_1.$$

In this case, the general solution of the equation can be described as

$$X = A_1^{+} C_1 + F_{A_1} L, \tag{3}$$

where $L \in \mathbf{C}^{m\times n}$ is an arbitrary matrix.

Let

$$A^{\mathrm{H}} = \left[A_2^{\mathrm{H}}, B_2\right]^{\mathrm{H}}, B^{\mathrm{H}} = \left[C_2^{\mathrm{H}}, D_2\right]^{\mathrm{H}}.$$

By Lemma 2, we know the matrix equations

$$A_2 Y = C_2, Y B_2 = D_2$$

has a Hermitian solution Y if and only if

$$AA^{+}B = B, AB^{\mathrm{H}} = BA^{\mathrm{H}},$$

in which case the general Hermitian solution of the equation can be expressed as

$$Y = A^{+}B + F_A(A^{+}B)^{\mathrm{H}} + F_A K F_A, \tag{4}$$

where $K \in \mathbf{C}^{p\times p}$ is an arbitrary Hermitian matrix.
Likewise, let

$$C^{\mathrm{H}} = \left[A_3^{\mathrm{H}}, B_3\right]^{\mathrm{H}}, D^{\mathrm{H}} = \left[C_3^{\mathrm{H}}, D_3\right]^{\mathrm{H}},$$

then the matrix equations

$$A_3 Z = C_3, Z B_3 = D_3$$

has a Hermitian solution Z if and only if

$$CC^{+}D = D, CD^{\mathrm{H}} = DC^{\mathrm{H}},$$

in which case the general Hermitian solution of the equation can be expressed as

$$Z = C^{+}D + F_C(C^{+}D)^{\mathrm{H}} + F_C J F_C, \tag{5}$$

where $J \in \mathbf{C}^{q\times q}$ is an arbitrary Hermitian matrix.

By substituting (3), (4) and (5) into the fourth equation of (1), we can obtain

$$M_1 L + L^{\mathrm{H}} M_1^{\mathrm{H}} + M_2 K M_2^{\mathrm{H}} + M_3 J M_3^{\mathrm{H}} = W_1, \tag{6}$$

where

$$M_1 = A_4 F_{A_1},\ M_2 = B_4 F_A,\ M_3 = C_4 F_C$$

and

$$\begin{aligned} &W_1 = D_4 - A_4 A_1^{+} C_1 - (A_4 A_1^{+} C_1)^{\mathrm{H}} - B_4 A^{+} B B_4^{\mathrm{H}} \\ - \quad &B_4 F_A (B_4 A^{+} B)^{\mathrm{H}} - C_4 C^{+} D C_4^{\mathrm{H}} - C_4 F_C (C_4 C^{+} D)^{\mathrm{H}}. \end{aligned}$$

According to Lemma 3, the equation of (6) with respect to L is solvable if and only if

$$G_1 K G_1^{\mathrm{H}} + G_2 J G_2^{\mathrm{H}} = W, \tag{7}$$

and the general solution of (6) with respect to L can be expressed as

$$\begin{aligned} L = &\tfrac{1}{2} M_1^{+} \tilde{D} + \tfrac{1}{2} M_1^{+} \tilde{D} E_{M_1} + U - \tfrac{1}{2} M_1^{+} M_1 U \\ &-\tfrac{1}{2} M_1^{+} U^{\mathrm{H}} M_1^{\mathrm{H}} - \tfrac{1}{2} M_1^{+} M_1 U E_{M_1}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} G_1 &= E_{M_1} M_2, \\ G_2 &= E_{M_1} M_3, \\ W &= E_{M_1} W_1 E_{M_1}, \\ \tilde{D} &= W_1 - M_2 K M_2^{\mathrm{H}} - M_3 J M_3^{\mathrm{H}} \end{aligned}$$

and U is an arbitrary matrix.

By (2), the equation of (7) can be equivalent written as

$$\Omega_1 P^{\mathrm{H}} K P \Omega_1^{\mathrm{H}} + \Omega_2 Q^{\mathrm{H}} J Q \Omega_2^{\mathrm{H}} = N^{-1} W (N^{-1})^{\mathrm{H}}. \tag{9}$$

Write

$$P^{\mathrm{H}} K P = [K_{ij}]_{3\times 3},\ K_{ij} = K_{ij}^{\mathrm{H}},\ i,j = 1,2,3, \tag{10}$$

$$Q^{\mathrm{H}} J Q = [J_{ij}]_{3\times 3},\ J_{ij} = J_{ij}^{\mathrm{H}},\ i,j = 1,2,3, \tag{11}$$

$$N^{-1} W (N^{-1})^{\mathrm{H}} = [W_{ij}]_{4\times 4},\ W_{ij} = W_{ij}^{\mathrm{H}},\ i,j = 1,2,3,4, \tag{12}$$

and the partitions of the matrices $P^{\mathrm{H}}KP, Q^{\mathrm{H}}JQ$ and $N^{-1}W(N^{-1})^{\mathrm{H}}$ are compatible with those of Ω_1 and Ω_2. Thus, from (9), we have

$$\begin{aligned} &\begin{bmatrix} K_{11} & K_{12}S_1 & 0 & 0 \\ S_1 K_{12}^{\mathrm{H}} & S_1 K_{22} S_1 + S_2 J_{22} S_2 & S_2 J_{23} & 0 \\ 0 & J_{23}^{\mathrm{H}} S_2 & J_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{12}^{\mathrm{H}} & W_{22} & W_{23} & W_{24} \\ W_{13}^{\mathrm{H}} & W_{23}^{\mathrm{H}} & W_{33} & W_{34} \\ W_{14}^{\mathrm{H}} & W_{24}^{\mathrm{H}} & W_{34}^{\mathrm{H}} & W_{44} \end{bmatrix}. \end{aligned} \tag{13}$$

By (13), we can get

$$\begin{aligned} &W_{13} = 0,\ W_{i4} = 0,\ i = 1,2,3,4, \\ &K_{11} = W_{11},\ K_{12} S_1 = W_{12}, \\ &S_1 K_{22} S_1 + S_2 J_{22} S_2 = W_{22}, \\ &J_{33} = W_{33},\ S_2 J_{23} = W_{23}. \end{aligned}$$

In summary of above discussion, we can easily obtain the following result.

Theorem 1: Suppose that

$$\begin{aligned} &A_1 \in \mathbf{C}^{a_1\times m},\ C_1 \in \mathbf{C}^{a_1\times n}, \\ &A_2,\ C_2 \in \mathbf{C}^{a_2\times p},\ B_2,\ D_2 \in \mathbf{C}^{p\times b_2}, \\ &A_3,\ C_3 \in \mathbf{C}^{a_3\times q},\ B_3,\ D_3 \in \mathbf{C}^{q\times b_3}, \\ &A_4 \in \mathbf{C}^{n\times m},\ B_4 \in \mathbf{C}^{n\times p},\ C_4 \in \mathbf{C}^{n\times q} \end{aligned}$$

and

$$D_4 = D_4^{\mathrm{H}} \in \mathbf{C}^{n\times n}.$$

Let

$$\begin{aligned} &A^{\mathrm{H}} = \left[A_2^{\mathrm{H}}, B_2\right]^{\mathrm{H}}, B^{\mathrm{H}} = \left[C_2^{\mathrm{H}}, D_2\right]^{\mathrm{H}}, \\ &C^{\mathrm{H}} = \left[A_3^{\mathrm{H}}, B_3\right]^{\mathrm{H}}, D^{\mathrm{H}} = \left[C_3^{\mathrm{H}}, D_3\right]^{\mathrm{H}}, \\ &M_1 = A_4 F_{A_1}, M_2 = B_4 F_A, M_3 = C_4 F_C, \\ &W_1 = D_4 - A_4 A_1^{+} C_1 - (A_4 A_1^{+} C_1)^{\mathrm{H}} - B_4 A^{+} B B_4^{\mathrm{H}} \\ &-B_4 F_A (B_4 A^{+} B)^{\mathrm{H}} - C_4 C^{+} D C_4^{\mathrm{H}} - C_4 F_C (C_4 C^{+} D)^{\mathrm{H}}, \\ &G_1 = E_{M_1} M_2, G_2 = E_{M_1} M_3, W = E_{M_1} W_1 E_{M_1}, \end{aligned}$$

the GSVD of (G_1, G_2) be given by (2) and $N^{-1}W(N^{-1})^{\mathrm{H}} = [W_{ij}]_{4\times 4}$ be given by (12). Then the equation of (1) has a solution (X, Y, Z) if and only if

$$\begin{aligned} &A_1 A_1^{+} C_1 = C_1,\ AA^{+}B = B,\ AB^{\mathrm{H}} = BA^{\mathrm{H}}, \\ &CC^{+}D = D,\ CD^{\mathrm{H}} = DC^{\mathrm{H}}, \\ &W_{13} = 0,\ W_{i4} = 0,\ i = 1,2,3,4, \end{aligned}$$

in which case, the general solution can be expressed as

$$\begin{aligned} X &= A_1^{+} C_1 + F_{A_1} L, \\ Y &= A^{+} B + F_A (A^{+} B)^{\mathrm{H}} + F_A K F_A, \\ Z &= C^{+} D + F_C (C^{+} D)^{\mathrm{H}} + F_C J F_C, \end{aligned}$$

where L is given by (8) with

$$\tilde{D} = W_1 - M_2 K M_2^{\mathrm{H}} - M_3 J M_3^{\mathrm{H}},$$

and

$$K = P \begin{bmatrix} W_{11} & W_{12}S_1^{-1} & K_{13} \\ S_1^{-1}W_{12}^{\mathrm{H}} & S_1^{-1}(W_{22} - S_2 J_{22} S_2)S_1^{-1} & K_{23} \\ K_{13}^{\mathrm{H}} & K_{23}^{\mathrm{H}} & K_{33} \end{bmatrix} P^{\mathrm{H}},$$

$$J = Q \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{12}^{\mathrm{H}} & J_{22} & S_2^{-1}W_{23} \\ J_{13}^{\mathrm{H}} & W_{23}^{\mathrm{H}}S_2^{-1} & W_{33} \end{bmatrix} Q^{\mathrm{H}},$$

and

$$U, K_{13}, K_{23}, K_{33} = K_{33}^{\mathrm{H}}, J_{11} = J_{11}^{\mathrm{H}}, J_{22} = J_{22}^{\mathrm{H}}, J_{12}, J_{13}$$

are all arbitrary matrices.

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