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A Further Study on the 4-Ordered Property of Some Chordal Ring Networks

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Abstract—Given a graph G. A cycle of G is a sequence of vertices of G such that the first and the last vertices are the same. A hamiltonian cycle of G is a cycle containing all vertices of G. The graph G is k-ordered (resp. k-ordered hamiltonian) if for any sequence of k distinct vertices of G, there exists a cycle (resp. hamiltonian cycle) in G containing these k vertices in the specified order. Obviously, any cycle in a graph is 1-ordered, 2-ordered and 3-ordered. Thus the study of any graph being k-ordered (resp. k-ordered hamiltonian) always starts with $k = 4$. Most studies about this topic work on graphs with no real applications. To our knowledge, the chordal ring families were the first one utilized as the underlying topology in interconnection networks and shown to be 4-ordered. Furthermore, based on our computer experimental results, it was conjectured that some of them are 4-ordered hamiltonian. In this paper, we intend to give some possible directions in proving the conjecture.

Keywords—Hamiltonian cycle, 4-ordered, Chordal rings, 3-regular.

I. Introduction

We consider finite, undirected and simple graphs only. Let $G = (V, E)$ be a graph, where V is the set of vertices of G and $E = \{(u, v) \mid u, v \in V\}$ is the set of edges of G, respectively. Let u, v be two vertices of G. If $e = (u, v) \in E$, then we say that the vertices u and v are *adjacent* in G. The *degree* of any vertex u is the number of distinct vertices adjacent to u. $N(u)$ denotes the set of vertices which are adjacent to u. If $|N(u)| = 3$ for any vertex u of G, then we call G a 3-*regular* graph. A *path* P between two vertices v_0 and v_k is represented by $P = \langle v_0, v_1, \ldots, v_k\rangle$, where each pair of consecutive vertices are connected by an edge. We also write the path $P = \langle v_0, v_1, \ldots, v_k\rangle$ as $\langle v_0, v_1, \ldots, v_i, Q, v_j, v_{j+1}, \ldots, v_k\rangle$, where Q denotes the path between v_i and v_j. A *hamiltonian path* between u and v, where u and v are two distinct vertices of G, is a path joining u to v that visits every vertex of G exactly once. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A *hamiltonian graph* is a graph with a hamiltonian cycle. A graph G is k-*ordered* (or k-*ordered hamiltonian*, resp.) if for any sequence of k distinct vertices of G, there exists a cycle (or a hamiltonian cycle, resp.) in G containing these k vertices in the specified order. Obviously, any cycle in a graph is 1-ordered, 2-ordered and 3-ordered. Thus the study of k-orderedness (or k-ordered hamiltonicity) of any graph always starts with $k = 4$.

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A graph $G = (V, E)$ is a k-*ordered hamiltonian-connected* graph if for any sequence of k vertices of G, there exists a hamiltonian path P between u and v such that P passes these vertices in the specified order. It can be seen that k-ordered hamiltonicity and k-ordered hamiltonian-connectedness do not imply each other.

The concept of k-orderedness and k-ordered hamiltonicity was first introduced by Ng and Schultz [2] in 1997. See [2], [3], [4], [5], [6] for example. In [2], the authors posed the question of the existence of 4-ordered 3-regular graphs other than the complete graph K_4 and the complete bipartite graph $K_{3,3}$. In [5], Meszaros answered the question by proving that the Petersen graph and the Heawood graph are non-bipartite, 4-ordered 3-regular graphs. Hsu et al. in [6] provided examples of bipartite non-vertex-transitive 4-ordered 3-regular graphs of order n for any sufficiently large even integer n. In 2013, Hung et al. further gave a complete classification of generalized Petersen graphs, $GP(n, 4)$, and showed the following theorems[7]:

Theorem 1. [7] Let $n \geq 9$. $GP(n, 4)$ is 4-ordered hamiltonian if and only if $n \in \{18, 19\}$ or $n \geq 21$.

Theorem 2. [7] Let $n \geq 9$. $GP(n, 4)$ is 4-ordered hamiltonian-connected if and only if $n \geq 18$.

Since Petersen graphs have been well-known and often provide examples or counterexamples for interesting graphic properties, the results of [7] and Theorems 1-2 might leave readers an impression that most 4-ordered graphs are 4-ordered hamiltonian, and most 4-ordered hamiltonian graphs are 4-ordered hamiltonian-connected. It could be misleading. Therefore, the authors intend to study this topic on graphs with real applications, and the chordal ring networks turn out to be a good subject. The chordal ring family has been adopted as the underlying topology of certain interconnection networks [8] and is studied for the real architecture for parallel and distributed systems due to the advantage of a built-in hamiltonian cycle, symmetry, easy routing and robustness. See [9] and its references.

This paper is organized as follows. In Section II, the formal definition of the chordal ring networks is given. We introduce some known results and techniques, which are related to the study of the 4-ordered properties. Then we propose two possible methods of proving the following conjecture in [1], and the difficulties it may encounter. Finally, a brief conclusion is given in Section III.

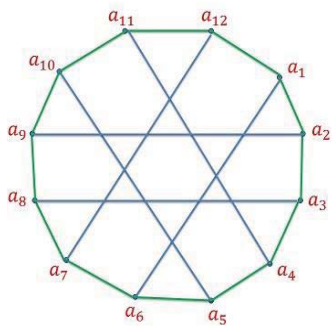

Fig. 1. $CR(12;1,5)$

Conjecture 1. [1] $CR(n;1,5)$ is a 4-ordered hamiltonian graph if $n=14$, $n=12k+2$ or $n=12k+10$ with $k\geq 2$.

II. About $CR(n;1,q)$ and its 4-ordered property

The chordal rings $CR(n;1,q)$, where n is an even integer with $n\geq 6$ and q an odd integer with $3\leq q\leq \frac{n}{2}$, is defined as follows. Let $G(V,E)=CR(n;1,q)$, where $V=\{a_1,a_2,\ldots,a_n\}$ and $E=\{(a_i,a_{(i+1)\mathbf{mod}n}):1\leq i\leq n\}\cup\{(a_i,a_{(i+q)\mathbf{mod}n}):i$ is odd and $1\leq i\leq n\}$. An illustration of $CR(12;1,5)$ is given in Fig. 1. Obviously, it is a 3-regular graph, has a built-in hamiltonian cycle $\langle a_1,a_2,\ldots,a_n,a_1\rangle$, and is vertex symmetric. The following theorems were proved in [1].

Theorem 3. [1] $CR(n;1,5)$ is 4-ordered for any even integer n with $n\geq 14$.

Theorem 4. [1] $CR(n;1,7)$ is 4-ordered for any even integer n with $n\geq 18$.

The first method we propose to prove Conjecture 1 is applying the similar technique for the above two theorems. Thus, before we present our possible methods of proving Conjecture 1, a short review of [1] about how the above two theorems are derived is necessary. In particular, the techniques required for Theorem 3 is recalled below. It can be observed that Theorem 3 is a combination of the following three theorems.

Theorem 5. [1] $CR(20+6k;1,5)$ is 4-ordered for any integer k with $k\geq 0$.

Theorem 6. [1] $CR(22+6k;1,5)$ is 4-ordered for any integer k with $k\geq 0$.

Theorem 7. [1] $CR(24+6k;1,)$ is 4-ordered for any integer k with $k\geq 0$.

Since Theorems 5-7 are proved with the similar method, here we only recall that of Theorem 5. Based on the computer experimental results, it was shown that $CR(20;1,5)$ is 4-ordered. Now, we define a function f, which maps R from $CR(20+6k;1,5)$ into $CR(26+6k;1,5)$ in the following

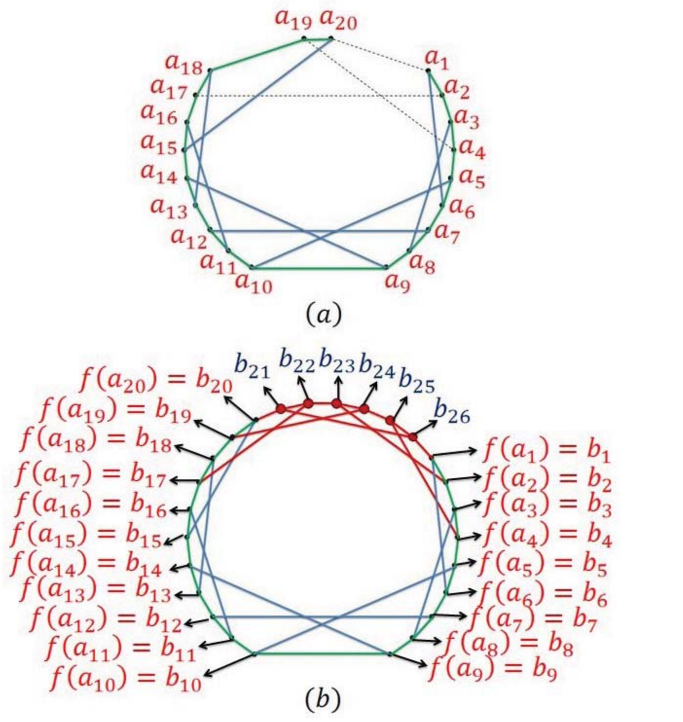

Fig. 2. From $CR(20;1,5)$ into $CR(26;1,5)$

way:
(1) If $a_i\in R\cap V(CR(20+6k;1,5))$, where $1\leq i\leq 20+6k$, then $f(a_i)=b_i$.

(2) If $(a_i,a_j)\in R\cap E(CR(20+6k;1,5))$, where $1\leq i,j\leq 20+6k$, then

$$f((a_i,a_j))=\begin{cases}(b_i,b_{i+1}) & 1\leq i\leq 19+6k, j=i+1;\\(b_i,b_{i+5}) & i=\text{ odd}, 1\leq i\leq 15+6k, j=i+5;\\\emptyset & \text{otherwise.}\end{cases}$$

It is easy to see that $f((a_{20+6k},a_1))=\emptyset$, $f((a_{17+6k},a_2))=\emptyset$ and $f((a_{19+6k},a_4))=\emptyset$. Therefore, $CR(26+6k;1,5)-f(CR(20+6k;1,5))$ consists of the vertex set $\{b_{21+6k},b_{22+6k},\ b_{23+6k},b_{24+6k},b_{25+6k},b_{26+6k}\}$ and the edge set $\{(b_{20+6k},b_{21+6k}),(b_{21+6k},b_{22+6k}),(b_{22+6k},b_{23+6k}),(b_{23+6k},b_{24+6k}),(b_{24+6k},b_{25+6k}),(b_{25+6k},b_{26+6k}),(b_{26+6k},b_1),(b_{17+6k},b_{22+6k}),(b_{19+6k},b_{24+6k}),(b_{21+6k},b_{26+6k}),(b_{23+6k},b_2),(b_{25+6k},b_4)\}$. Fig. 2 gives an illustration, in which f maps R from $CR(20;1,5)$ into $CR(26;1,5)$. We can see the following things happen.

(1) $f(a_i)=b_i$ for $1\leq i\leq 20$, denoted by black vertices on both graphs.

(2) $f((a_i,a_{i+1}))=(b_i,b_{i+1})$ for $1\leq i\leq 19$, denoted by green edges on both graphs.

(3) $f((a_i,a_{i+5}))=(b_i,b_{i+5})$ for i is odd with $1\leq i\leq 15$, denoted by blue edges on both graphs.

(4) $f((a_{20},a_1))=\emptyset$,$f((a_{17},a_2))=\emptyset$ and $f((a_{19},a_4))=\emptyset$,

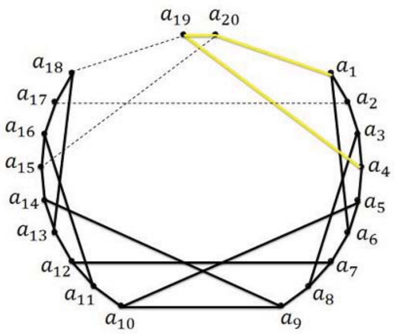

Fig. 3. $CR(20; 1, 5)$

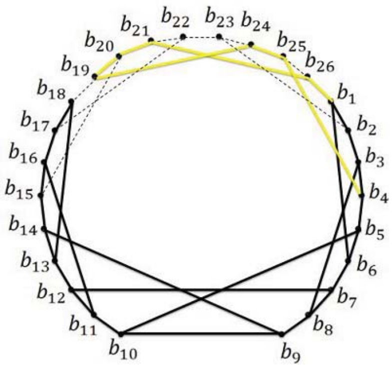

Fig. 4. $CR(26; 1, 5)$

denoted by dashed edges on $CR(20; 1, 5)$.

(5) $CR(26; 1, 5) - f(CR(20; 1, 5))$ consists of the vertex set $\{b_{21}, b_{22}, b_{23}, b_{24}, b_{25}, b_{26}\}$ and edge set $\{(b_{20}, b_{21})$, (b_{21}, b_{22}), (b_{22}, b_{23}), (b_{23}, b_{24}), (b_{24}, b_{25}), (b_{25}, b_{26}), (b_{26}, b_1), (b_{17}, b_{22}), (b_{19}, b_{24}), (b_{21}, b_{26}), (b_{23}, b_2), $(b_{25}, b_4)\}$.

As an example, we show the construction of the required cycle in $CR(26; 1, 5)$ using the known cycle of $CR(20; 1, 5)$. There are 20 vertices $a_1, a_2, ..., a_{20}$ in $CR(20; 1, 5)$, and 26 vertices $b_1, b_2, ..., b_{26}$ in $CR(26; 1, 5)$. To prove the theorem, we do case studies by considering different situations. Take $G = CR(26; 1, 5)$. Let x_1, x_2, x_3 and x_4 be four arbitrary vertices of G. We want to construct a cycle C in G that visits $x_i{}' s$ in the given order. Note that we can always find at least one set of six consecutive vertices, denoted by $S = \{b_i, b_{i+1}, b_{i+2}, ..., b_{i+5}\}$, such that $S \cap \{x_1, x_2, x_3, x_4\} = \phi$. Without loss of generality, let $x_1 = b_1$ and $S = \{b_{21}, b_{22}, ..., b_{26}\}$. Removing the vertices of S and all edges adjacent to S in G, we obtain a subgraph of $CR(20; 1, 5)$. Obviously, $S \cap f(CR(20; 1, 5)) = \phi$. Note that $CR(20; 1, 5)$ is 4-ordered and hence contains a cycle that visits $x_i{}' s$ in the given order, denoted by C'. We will obtain C by embedding $CR(20; 1, 5)$ into $CR(26; 1, 5)$ and rerouting the cycle C'. One of the possible rerouting cases are illustrated in Fig. 3 and 4, where we reroute a path in $CR(20; 1, 5)$ to obtain a path in $CR(26; 1, 5)$ without changing the corresponding location of any other vertex in $CR(26; 1, 5)$.

Once such an embedding and rerouting method is realized,

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0
0	5	6	11	12	17	18	23	24	3	4	9	10	15	16	21	22	1	2	7	8	13	14	19	20	25	0
0	5	6	7	2	3	4	9	8	13	12	11	10	15	14	19	18	17	16	21	20	25	24	23	22	1	0

Fig. 5. Simulation Results

we can talk about the general construction method as follows. Given any four vertices in $G = CR(n; 1, 5)$, denoted by a_1, a_i, a_j, a_k with $1 < i < j < k$, we want to construct a hamiltonian cycle which passes through these four vertices in any specified order. The computer experimental results show that the conjecture holds for $k = 2, 3, 4$ by giving the required hamiltonian cycles concretely. Note that when $k = 4$, the total number of vertices of G is 50 or 58, and we want to verify the conjecture for $k \geq 5$. It is easy to see that there exists a set S of at least 12 consecutive vertices on G such that $S \cap \{a_1, a_i, a_j, a_k\} = \emptyset$. Therefore, we can embed the graph $G = CR(n; 1, 5)$ into $\bar{G} = CR(n + 12; 1, 5)$ by a proper function f, which will map the known hamiltonian cycle C in G into $\bar{G}$ as well. Since there are additional 12 vertices in $\bar{G}$ not included in the cycle C, we must develop a routing algorithm that relies on C to keep the original order of $\{a_1, a_i, a_j, a_k\}$ and extends C to go through the 12 vertices of $\bar{G}$. Compared with the work in [1], it becomes rather complicated for the 4-ordered hamiltonicity. Since some of the paths on C has to be broken in order for rerouting in $\bar{G}$, the requirement that all of these extra 12 vertices must be covered makes the routing scheme very difficult. In many cases, one must give up the known hamiltonian cycle C provided by the computer and reconstruct another hamiltonian cycle C' in G for the purpose of "extension". The working load of case-study analysis as in [1] could be doubled or even tripled.

The second method we propose requires some combinatorial knowledge. Since $CR(n; 1, q)$ is vertex symmetric, without loss of generality, we can always let a_1 be one of the four given vertices which need to be visited in order and let the other three vertices be $x = a_i, y = a_j$ and $z = a_k$, where $1 < i < j < k$. It is natural to ask the question regarding the number of hamiltonian cycles required to show the 4-ordered hamiltonicity. Given any four vertices, since they lie on a circle, if we can find three hamiltonian cycles on which these four vertices appear as $a_1 \to x \to y \to z \to a_1$, $a_1 \to y \to x \to z \to a_1$, and $a_1 \to x \to z \to y \to a_1$, then it is done. In fact, theoretically, three hamiltonian cycles which are "independent" of each other should be enough. However, the following example tells us that the construction for such three hamiltonian cycles could be very difficult.

In Fig. 5, any vertex of $G = CR(26; 1, 5)$ is labeled by an integer i with $0 \leq i \leq 25$ for simplicity. Any (i, j) is an edge of G if and only if $j = (i \pm 1) \bmod 26$ or $j = (i + 5) \bmod 26$. Suppose that the given four vertices are $0, 1, 2, 3$, it is observed that the three hamiltonian cycles presented in Fig. 2 provide the required cycles with $0, 1, 2, 3$ in any specified order. However, if the given four vertices in G are $0, 12, 17, 23$, the three hamiltonian cycles in Fig. 5 gives no help other than the natural built-in hamiltonian cycle of the chordal ring graphs. Thus the following two questions arise.

Q.1. Is it possible to construct three and only three hamiltonian cycles in $CR(n;1,5)$, denoted by C_1, C_2 and C_3, which are "totally independent" so that for any given four vertices with a specific order, one could find the required one among them? Does the structure of $CR(n;1,5)$ and its algebraic pattern, after excluding the repetitions due to symmetry, hinder the optimum to occur?

Q.2. If the answer to the first question is yes, then how can we find these three hamiltonian cycles? If the answer is no, then what's the least number of hamiltonian cycles we need to include all possible 4-ordered cases? Once again, if we only need k hamiltonian cycles for $CR(n;1,5)$ for 4-ordered hamiltonicity, where are these k ones? A concrete construction scheme must be developed.

III. Conclusion

In this paper, we continue the study of 4-ordered properties of the chordal ring networks. A rigorous proof of the 4-ordered hamiltonicity of any graph family has been shown a difficult work (see [6], [7] for example). For $CR(n;1,5)$, Conjecture 1 has been proposed based on computer experimental results. We not only propose two possible techniques for verifying the conjecture, but also present the corresponding complexities. Finally, two interesting topics following this discussion are raised for future studies.

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