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New Fourth Order Explicit Group Method in the Solution of the Helmholtz Equation

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Abstract**—**In this paper, the formulation of a new group explicit method with a fourth order accuracy is described in solving the two dimensional Helmholtz equation. The formulation is based on the nine-point fourth order compact finite difference approximation formula. The complexity analysis of the developed scheme is also presented. Several numerical experiments were conducted to test the feasibility of the developed scheme. Comparisons with other existing schemes will be reported and discussed. Preliminary results indicate that this method is a viable alternative high accuracy solver to the Helmholtz equation.

Keywords**—**Explicit group method, finite difference, Helmholtz equation, five-point formula, nine-point formula.

I. Introduction

We consider the two dimensional Helmholtz equation of the form

$$u_{xx} + u_{yy} + k^2 u = f\left(x, y\right),\ (x, y) \in \Omega \qquad (1)$$

where the solution domain $\Omega = (0,1)\mathrm{x}(0,1)$ with Dirichlet conditions defined at the boundary. The solution is $u(x,y)$, k is known as a wave number, and the function f together with u are assumed to be sufficiently smooth and have necessary continuous partial derivatives. This equation governs some problems in physical phenomena such as water wave propagation and membrane vibration [4]. The solution domain is discretized uniformly in the x and y directions so that the mesh size is $h=1/n$, where $x_i = ih$, $y_j = jh$ $\left(i, j = 0,1,2,\cdots,n\right)$. The notation u_{ij} is used to represent the computed solution $u(x_i, y_j)$. Recently, several new point and group schemes derived from the standard and rotated five-point stencils have been developed in solving this type of problem [1]-[3]. In [2], for example, a half-sweep point iterative method is derived in solving model problem (1) where this method is found to be have better convergence rates than the normal full-sweep iterative scheme due to the lesser computing complexity of the former. Group iterative schemes were formulated in [1] and [3] to solve the same equation where these new methods were proven to have even better convergence rates than the point scheme in [2]. However, all these developed schemes are of second order accuracy.

In this paper, a new group iterative scheme with a higher order accuracy will be developed in solving (1). This four-point explicit group method is formulated by using the compact nine-point finite difference formula with fourth order accuracy. Section II will describe the derivation of the proposed scheme, followed by the complexity analysis of the scheme in Section III. Section IV will report some numerical experiments implemented on the proposed scheme. Concluding remarks are given in Section V.

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II. Formulation of the Group Method

A. Explicit Group $O(h^2)$

The standard second order central difference can be written as

$$\delta_x^2 u_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2},\ \ \delta_y^2 u_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} \qquad (2)$$

If (1) is approximated by the standard five-point central difference scheme (2), the following is obtained

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} + (k^2h^2 - 4)u_{i,j} = h^2 f_{i,j}\,. \qquad (3)$$

This scheme has a truncation error of $O(h^2)$. Based on this approximation, the second order explicit group scheme was derived by [1] as:

$$\begin{bmatrix} u_{i,j} \\ u_{i+1,j} \\ u_{i+1,j+1} \\ u_{i,j+1} \end{bmatrix} = \begin{bmatrix} p(p^2-2) & p^2 & 2p & p^2 \\ p^2 & p(p^2-2) & p^2 & 2p \\ 2p & p^2 & p(p^2-2) & p^2 \\ p^2 & 2p & p^2 & p(p^2-2) \end{bmatrix} \mathrm{x} \frac{1}{p^2(p^2-4)} \begin{bmatrix} u_{i-1,j} + u_{i,j-1} - h^2 f_{i,j} \\ u_{i+2,j} + u_{i+1,j-1} - h^2 f_{i+1,j} \\ u_{i+2,j+1} + u_{i+1,j+2} - h^2 f_{i+1,j+1} \\ u_{i-1,j+1} + u_{i,j+2} - h^2 f_{i,j+1} \end{bmatrix} \qquad (4)$$

where $p = (4 - k^2h^2)$. Akhir et al. [1] generate the iterations in groups of four until a certain convergence criteria is met. They found that this EG $O(h^2)$ is more superior in terms of execution timings than the standard five-point scheme (3) but with the same order of accuracy.

B. Explicit Group O(h⁴)

To design a higher order scheme, the Taylor series expansion is used to obtain the following formulas:

$$\delta_x^2 u_{i,j} = u_{xx} + \frac{h^2}{12} u_x^4 + \frac{h^4}{360} u_x^6 + O(h^6) \tag{5}$$

$$\delta_y^2 u_{i,j} = u_{yy} + \frac{h^2}{12} u_y^4 + \frac{h^4}{360} u_y^6 + O(h^6) \tag{6}$$

which can be re-arranged to become [5]

$$u_{xx} = \left(1 + \frac{h^2}{12}\delta_x^2\right)^{-1} \delta_x^2 u_{i,j} \tag{7}$$

$$u_{yy} = \left(1 + \frac{h^2}{12}\delta_y^2\right)^{-1} \delta_y^2 u_{i,j}\,. \tag{8}$$

By employing (7) and (8) into (1), and rearranging, we have the following

$$\begin{aligned} & u_{i+1,j+1} + u_{i+1,j-1} + u_{i,-1j+1} + u_{i-1,j-1} + \\ & \left(4 + \frac{k^2h^2}{2}\right)(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}) + (4k^2h^2 - 20)u_{i,j} \\ & = \frac{h^2}{2}\left(8f_{i,j} + f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1}\right) \end{aligned} \tag{9}$$

which is a fourth order compact approximation to (1). The computational molecule of this approximation is of the following form:

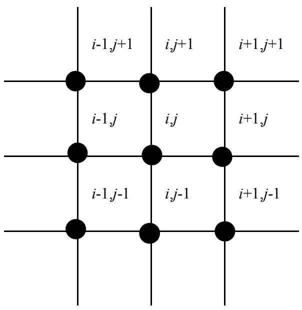

Fig. 1 Computational molecule of approximation (9)

The compact nine-point iterative scheme may be formulated based on (9) where iterations are generated on the points in the whole solution domain until convergence is achieved. To construct the proposed explicit group scheme, we apply (9) to groups of four points in the solution domain which will result in the following (4x4) system of equations:

$$\begin{bmatrix} p & q & q & -1 \\ q & p & -1 & q \\ q & -1 & p & q \\ -1 & q & q & p \end{bmatrix} \begin{bmatrix} u_{i,j} \\ u_{i+1,j} \\ u_{i,j+1} \\ u_{i+1,j+1} \end{bmatrix} = \begin{bmatrix} u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} - \frac{h^2}{2}F_1 \\ u_{i+2,j+1} + u_{i+2,j-1} + u_{i,j+1} - \frac{h^2}{2}F_2 \\ u_{i+1,j+2} + u_{i-1,j} + u_{i-1,j+2} - \frac{h^2}{2}F_3 \\ u_{i+2,j+2} + u_{i+2,j} + u_{i,j+2} - \frac{h^2}{2}F_4 \end{bmatrix} - q \begin{bmatrix} u_{i-1,j} + u_{i,j-1} \\ u_{i+2,j} + u_{i+1,j-1} \\ u_{i,j+2} + u_{i-1,j+1} \\ u_{i+1,j+2} + u_{i+2,j+1} \end{bmatrix} \tag{10}$$

where

$$\begin{aligned} F_1 &= 8f_{i,j} + f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} \\ F_2 &= 8f_{i+1,j} + f_{i+1,j+1} + f_{i+2,j} + f_{i+1,j-1} + f_{i,j} \\ F_3 &= 8f_{i,j+1} + f_{i,j+2} + f_{i+1,j+1} + f_{i,j} + f_{i-1,j+1} \\ F_4 &= 8f_{i+1,j+1} + f_{i+1,j+2} + f_{i+2,j+1} + f_{i+1,j} + f_{i,j+1} \end{aligned}$$

$p = (20 - 4k^2h^2)$, and $q = -\left(4 + \frac{k^2h^2}{2}\right)$. The matrix at the left hand side of the system (10) can be inverted which transform the system into an explicit form:

$$\begin{bmatrix} u_{i,j} \\ u_{i+1,j} \\ u_{i,j+1} \\ u_{i+1,j+1} \end{bmatrix} = \frac{1}{p^4 + \left(-4q^2 - 2\right)p^2 - 8q^2p + \left(-4q^2 + 1\right)} \begin{bmatrix} b & c & c & d \\ c & b & d & c \\ c & d & b & c \\ d & c & c & b \end{bmatrix} F \tag{11}$$

where

$$F = \begin{bmatrix} u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} - \frac{h^2}{2}F_1 \\ u_{i+2,j+1} + u_{i+2,j-1} + u_{i,j+1} - \frac{h^2}{2}F_2 \\ u_{i+1,j+2} + u_{i-1,j} + u_{i-1,j+2} - \frac{h^2}{2}F_3 \\ u_{i+2,j+2} + u_{i+2,j} + u_{i,j+2} - \frac{h^2}{2}F_4 \end{bmatrix} - q \begin{bmatrix} u_{i-1,j} + u_{i,j-1} \\ u_{i+2,j} + u_{i+1,j-1} \\ u_{i,j+2} + u_{i-1,j+1} \\ u_{i+1,j+2} + u_{i+2,j+1} \end{bmatrix}$$

$b = \left(p^2 - p - 2q^2\right)\left(p+1\right)$, $c = -q\left(p+1\right)^2$ and

$d = \left(p + \left(2q^2 - 1\right)\right)\left(p+1\right)$.

The truncation error of this scheme is of *O(h⁴)*.

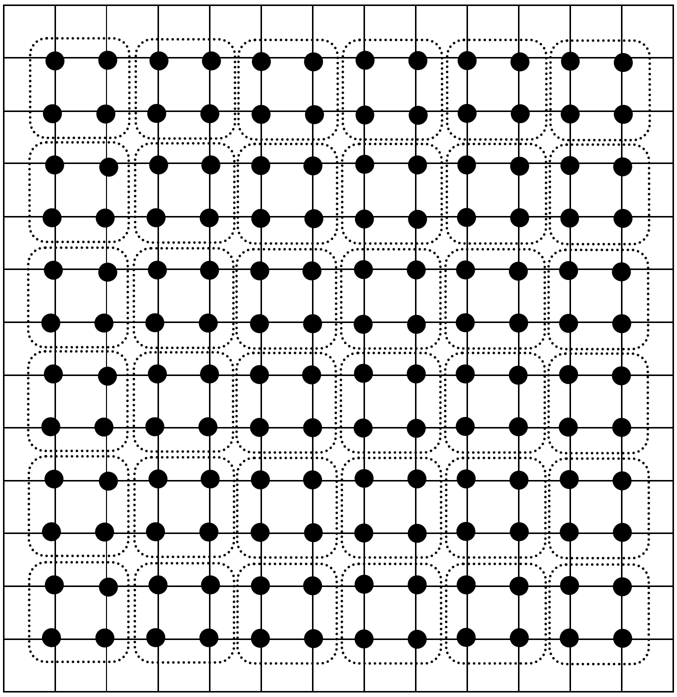

Fig. 2 Grouping of points for the group method (n=13)

All the points in the solution domain are divided in groups of four as shown in Fig. 2 for the case n = 13. Using (11), iterations on these groups of points are generated until a certain convergence criteria is met. In summary, we describe the fourth order explicit group method as follows:

1. Divide the solution domain into groups of four points as shown in Fig. 2 (for the case n=13).
2. Iterate on the points ● in each group using (11) with preferred smoother (e.g. Multigrid, Successive OverRelaxation, etc)
3. Check the convergence. If the solutions converge, terminate the iterations. Otherwise, repeat step 2.

III. Complexity Analysis

Assume that the solution domain is discretized with integer n, then the number of internal mesh points is given by m^2 where m=n-1. If n is even, then there will be ungrouped points near the upper/right boundaries. These type of points will be smoothed using (9). Table I lists the number of points for the internal mesh points for the proposed explicit group method as well as for the existing second order explicit group method due to [1], the nine-point compact and the five-point method. The computational complexity of the algorithm is based on the number of arithmetic operations performed at each iteration, i.e. the operation involving addition/subtraction $(+/-)$ and multiplication/division $(\times/\div)$ performed in each iteration, excluding the convergence test. The execution time required to perform an arithmetic operation is assumed to be the same or almost the same. All the points in the solution domain are involved in the iterative process, i.e. m^2. However, for the group method, it will require $(m-1)^2$ interior points and $(2m-1)$ points near the upper/right boundary if n is even.

TABLE I
Number of Points Involved in the Iterations

Types of points	EG $O(h^2)$	EG $O(h^4)$	Compact Nine-Point Scheme	Standard Five Point Scheme
Iterative grouped points	$4\times\left\lfloor\frac{m^2}{2}\right\rfloor$	$4\times\left\lfloor\frac{m^2}{2}\right\rfloor$	-	-
Iterative ungrouped points	$m^2-4\times\left\lfloor\frac{m^2}{2}\right\rfloor$	$m^2-4\times\left\lfloor\frac{m^2}{2}\right\rfloor$	-	-
Total internal points	m^2	m^2	m^2	m^2

The number of arithmetic operations required for each iteration for the point and group methods in terms of m are recorded in Table II. Note that, ceiling function is defined as $\lceil x\rceil = \min\{n \in \mathbb{Z} | n \geq x\}$, e.g. $\lceil 4.9\rceil = 5$, $\lceil -4.9\rceil = -4$, and floor function is defined as $\lfloor x\rfloor = max\{n \in \mathbb{Z} | n \leq x\}$, e.g. $\lfloor 4.9\rfloor = 4$, $\lfloor -4.9\rfloor = -5$.

TABLE II
Arithmetic Operations of the Methods under Study

Method	Per Iteration	
	+/-	×/÷
Standard Five Point	$4m^2$	m^2
EG $O(h^2)$	$4\times\left[m^2-4\times\left\lfloor\frac{m}{2}\right\rfloor^2\right]+11\times\left\lfloor\frac{m}{2}\right\rfloor^2$	$\left[m^2-4\times\left\lfloor\frac{m}{2}\right\rfloor^2\right]+5\times\left\lfloor\frac{m}{2}\right\rfloor^2$
Compact Nine Point	$8m^2$	$2m^2$
EG $O(h^4)$	$8\times\left[m^2-4\times\left\lfloor\frac{m}{2}\right\rfloor^2\right]+23\times\left\lfloor\frac{m}{2}\right\rfloor^2$	$2\times\left[m^2-4\times\left\lfloor\frac{m}{2}\right\rfloor^2\right]+9\times\left\lfloor\frac{m}{2}\right\rfloor^2$

IV. Numerical Experiments

To test the feasibility of the proposed method, we conduct numerical experiments in solving the following model problem [4]:

$$u_{xx}+u_{yy}+k^2u=\left(k^2-2\pi^2\right)\sin\left(\pi x\right)\sin\left(\pi y\right) \qquad (12)$$

where the solution domain is $\Omega=(0,1)\text{x}(0,1)$ with Dirichlet conditions defined at the boundary satisfying the exact solution $u(x,y)=\sin(\pi x)\sin(\pi y)$. All the experiments are conducted using the programming tool C++ on HP Mini 210-1000 with Windows 7 Starter Edition, processor type is Intel® Atom^{TM} CPU N450 @ 1.66GHz 1.67GHz, with installed memory (RAM) of 1GB and 32-bit Operating System type.

Throughout this experiment, all the algorithms are implemented using different grid sizes of 8, 16, 32, 64 and 128, with the values of k randomly chosen. For the iteration process, the multigrid technique was used as the smoother. Several parameters were measured, i.e. the execution times (in seconds), the number of iterations (Iter), the maximum absolute errors over the discrete grid, and the estimated order of accuracy. The maximum errors are taken as

$max\left|u_{i,j}^{(k+1)} - u_{i,j}^{(k)}\right| < \varepsilon$, where the tolerance $\varepsilon = 10^{-13}$. The estimated order of accuracy for each scheme can be computed for different grid size. Consider two mesh sizes Δ^H and Δ^h on Ω^H and Ω^h respectively. We denote the maximum absolute errors of these two grids as $Error^H$ and $Error^h$ respectively. Suppose the order of accuracy is A, then [5]

$$\frac{\left(\Delta^H\right)^A}{\left(\Delta^h\right)^A} = \frac{Error^H}{Error^h}. \tag{13}$$

So the order of accuracy can be estimated as

$$A = \frac{\log \dfrac{Error^H}{Error^h}}{\log \dfrac{\Delta^H}{\Delta^h}}. \tag{14}$$

For comparison purposes, we also run the programs for the existing five-point iterative scheme (3), nine-point iterative scheme (9) and the second order explicit group iterative scheme due to [1] (4). Table III displays the performances of the implemented numerical schemes.

TABLE III
PERFORMANCE COMPARISON BETWEEN METHODS

Methods	***n***	***Iter***	***Time (secs)***	***Max error***	***Order of Accuracy (A)***
Second Order (Standard Five Point)	8	70	0.02058	1.6295715e-02	-
	16	198	0.17208	4.0403450e-03	2.0
	32	582	1.67167	1.0080067e-03	2.0
	64	1795	21.045181	2.5187200e-04	2.0
	128	5739	289.70744	6.2959899e-05	2.0
Second Order EG [1]	8	51	0.01427	1.6295715e-02	-
	16	130	0.09486	4.0403450e-03	2.0
	32	365	0.99144	1.0080067e-03	2.0
	64	1093	11.64828	2.5187200e-04	2.0
	128	3432	161.12459	6.2959899e-05	2.0
Compact Fourth Order (Nine Point)	8	63	0.02142	8.2061725e-05	-
	16	175	0.19606	5.1660378e-06	4.0
	32	509	2.04716	3.2344042e-07	4.0
	64	1557	26.91834	2.0223900e-08	4.0
	128	4948	343.47874	1.2644070e-09	4.0
Fourth Order EG	8	57	0.02056	8.2061725e-05	-
	16	129	0.12877	5.1660378e-06	4.0
	32	352	1.27219	3.2344040e-07	4.0
	64	1040	15.28502	2.0223875e-08	4.0
	128	3242	209.53039	1.26432209e-09	4.0

The number of iterations for both five-point and compact nine-point iterative methods are almost the same, but the execution times for the compact nine-point stencil become longer because there are more points involved in the computational molecule, and also full weighted restriction operator is used during the multigrid smoother process for the this scheme, compared to the half weighted restriction operator used for the standard five-point stencil. However, the nine-point scheme gives fourth order approximation to the solutions compared to five-point scheme.

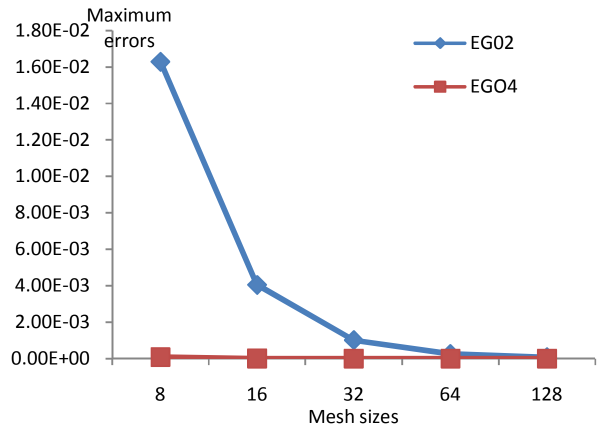

Fig. 3 Maximum errors for EG $O(h^2)$ and EG $O(h^4)$ for different mesh sizes

Amongst the group methods, the newly developed EG $O(h^4)$ produces more accurate solutions compared to the existing second order accurate solutions of EG $O(h^2)$ when the mesh is finer as shown in Fig. 3. But the complexity of EGO4 is larger than EG02 such that it requires slightly more CPU times than the latter. However the differences in timings are not very large as the grid sizes increase. It can be observed that EGO2 requires about 77% of the timings required by EGO4. Between the pointwise and the group methods, the group methods are more superior in terms of computing effort where they require only almost half of the execution times compared to that of their point iterative counterparts as shown in Fig. 4.

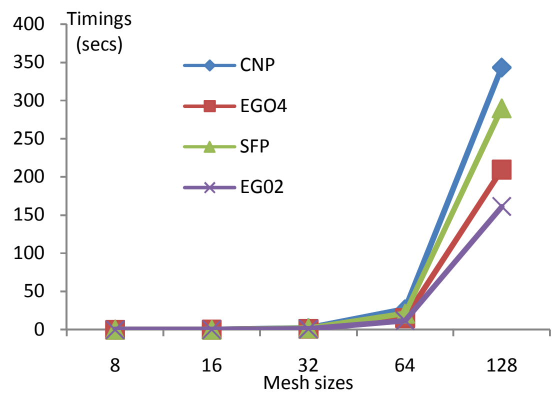

Fig. 4 Execution timings (in secs) for the compact nine-point (CNP) the explicit group $O(h^4)$ (EGO4), the standard five-point (SFP) and the second order explicit group $O(h^2)$ (EGO2)

Tables IV and V tabulate the total computing efforts between the pointwise and the group methods in terms of operation counts. These values were computed by combining the number of operations in Table II with the number of iterations obtained in the experiments by each scheme. From the experimental results, we can clearly see that the experimental timings of the iterative methods are in agreement with the complexity analysis.

TABLE IV
COMPUTING EFFORTS FOR STANDARD FIVE-POINT AND EXPLICIT GROUP OF ORDER 2 [1]

n	**Methods**			
	Standard Five Point		**EG $O(h^2)$ [1]**	
	Iter	*Operation count*	*Iter*	*Operation count*
8	70	20580	51	10659
16	198	267300	130	120770
32	582	3355812	365	1425325
64	1795	42746130	1093	17489093
128	5739	555385986	3432	222287208

TABLE V
COMPUTING EFFORTS FOR COMPACT NINE-POINT AND EXPLICIT GROUP OF ORDER 4

n	**Methods**			
	Compact Nine Point		**EG $O(h^4)$**	
	Iter	*Operation count*	*Iter*	*Operation count*
8	63	30870	57	23826
16	175	393750	129	239682
32	509	4891490	352	2749120
64	1557	61797330	1040	33282080
128	4948	798062920	3242	419962196

V. CONCLUSION

In this work, we have developed an alternative fourth order numerical scheme using a specific group construction in solving the two dimensional Helmholtz equation. The originality of this method lies in the selection of the grouping strategy which led to an explicit nature of the approximation formula. Numerical results show that the new method is able to solve the Helmholtz equation with better accuracy compared to the second order group method introduced in [1] without adding too much computing costs. The new scheme also requires lesser computing times than its point iterative counterpart, i.e. the compact nine-point scheme, which is due to the former's lower computational complexity. For future work, we foresee that this group method can also be extended to solve time dependent partial differential equations such as the 2D diffusion or convection-diffusion equations.

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