Fekete-Szegö Problem for Subclasses of Analytic Functions Defined by New Integral Operator

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Abstract—The author introduced the integral operator , by using this operator a new subclasses of analytic functions are introduced. For these classes, several Fekete-Szegö type coefficient inequalities are obtained.

Keywords—Integral operator, Fekete-Szegö inequalities, Analytic functions.

I. INTRODUCTION AND DEFINITION

Let \( A \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the open unite disk \( U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \).

Also let \( S \) denote the subclasses of \( A \) consisting of functions which are univalent in \( U \).

In [2] Fekete and Szegö proved a noticeable result that the estimate

\[
|a_3 - \mu a_2^2| \leq 1 + 2\exp\left(\frac{-2\mu}{1 - \mu}\right)
\]

holds for \( f \in S \) and for \( 0 \leq \mu \leq 1 \). This inequality is sharp for each \( \mu \). The coefficient functional

\[
\phi_\mu(f) = a_3 - \mu a_2^2 = \frac{1}{6} \left( f''(0) - \frac{3\mu}{2} (f''(0))^2 \right)
\]

on \( f \in A \) represents various geometric quantities as well as in the sense that this behaves well with respect to the rotation, namely

\[
\phi_\mu(e^{-i\theta} f(e^{i\theta} z)) = e^{2i\theta} \phi_\mu(f), \quad (\theta \in \mathbb{R}).
\]

In fact, other than the simplest case when

\[
\phi_0(f) = a_3,
\]

we have several important ones. For example,

\[
\phi_1(f) = a_3 - a_2^2.
\]

II. RESULTS

Recently, in [1] the author introduced a certain integral operator \( \mathcal{I} = e^\delta \) defined by :

\[
\mathcal{I}_\delta f(z) = \frac{(1 + e)^\delta}{\Gamma(\delta)} \int_0^1 t^{e-1} (\log 1/t)^{\delta-1} f(tz)dt,
\]

where \( c > 0, \delta > 1 \) and \( z \in U \).

Thus, it is quite natural to ask about inequalities for \( \phi_\mu \) corresponding to subclasses of \( S \). This is called Fekete-Szegö problem. Actually many authors have considered this problem for typical classes of univalent functions.

represent \( S_f(0)/6 \), where \( S_f \) denotes the Schwarzian derivative

\[
S_f(z) = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.
\]

Moreover, the first two non-trivial coefficients of the \( n \)-th root transform

\[
(f(z^n))^\frac{1}{n} = z + c_{n+1} z^{n+1} + c_{2n+1} z^{2n+1} + \ldots
\]

of \( f \) with the power series (1), are written by

\[
c_{n+1} = \frac{a_2}{n}
\]

and

\[
c_{2n+1} = a_3 \frac{n(n-1)a_2^2}{2n^2}
\]

so that

\[
a_3 - \mu a_2^2 = n(c_{2n+1} - \lambda c_{n+1}^2),
\]

where

\[
\lambda = \mu n + n - \frac{1}{2}.
\]

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\]

where \( c > 0, \delta > 1 \) and \( z \in U \).

We also note that the operator \( \mathcal{I}_\delta f(z) \) defined by (1) can be expressed by the series expansion as following:
In particular, we have starlike and convex function classes, 
\( S_{c,0}(1) = S^* \) and \( C_{c,0}(1) = C \), respectively.

We denote by \( P \) a class of the analytic functions in \( U \) with

\[ p(0) = 1 \text{ and } \Re\{p(z)\} > 0. \]

To prove our results, we need the following Lemmas considered by Duren [8] Ravichandran et al. [9].

**Lemma 1:** [8] Let \( p \in P \) with \( p(z) = 1 + c_1z + c_2z^2 + \ldots \). Then

\[ |c_n| \leq 2, \quad (n \geq 1). \]

**Lemma 2:** [9] Let \( p \in P \) with \( p(z) = 1 + c_1z + c_2z^2 + \ldots \).

Then for any complex number \( \gamma \)

\[ |c_2 - \gamma^2c_1^2| \leq 2 \max\{1, |2\gamma - 1|\}, \]

and the result is sharp for the functions given by

\[ p(z) = \frac{1 + z^2}{1 - z^2}, \quad p(z) = \frac{1 + z}{1 - z}. \]

**Lemma 3:** [8] Let \( p \in P \) with \( p(z) = 1 + c_1z + c_2z^2 + \ldots \).

Then

\[ |c_2 - \frac{1}{2}\lambda c_1^2| \leq 2 + \frac{1}{2}(|\lambda - 1| - 1)|c_1|^2. \]

II. MAIN RESULTS

**Theorem 1:** Let \( c, \delta \geq 0; b \in \mathbb{C}\setminus\{0\} \). If \( f \in S_{c,\delta}(b) \), then

\[ |a_2| \leq 2|b| \left(\frac{c + 2}{c + 1}\right)^\delta, \]

\[ |a_3| \leq |b| \left(\frac{c + 3}{c + 1}\right)^\delta \max\{1, |1 + 2b|\}, \]

and

\[ |a_3 - \frac{1}{2}\left(\frac{(c + 1)(c + 3)}{(c + 2)^2}\right)^\delta a_2| \leq |b| \left(\frac{c + 3}{c + 1}\right)^\delta. \]

**Proof.** Denote

\[ f \in C_{c,\delta}(b) \iff zf' \in S_{c,\delta}(b). \]

\[ \mathcal{I}_c^\delta f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1 + c}{k + c}\right)^\delta a_k z^k. \]
Then by (3), we can write
\[ A_2 = \left( \frac{c+1}{c+2} \right)^\delta a_2, \quad A_3 = \left( \frac{c+1}{c+3} \right)^\delta a_3. \] (10)

by the definition of the class \( S_{c,\delta}(b) \), there exists \( p \in P \) such that:
\[
1 + \frac{1}{b} \left( \frac{z(I_b^z f(z))'}{I_b^z f(z)} - 1 \right) = p(z),
\]
\[
\frac{z(I_b^z f(z))'}{I_b^z f(z)} = 1 - b + bp(z),
\]
so that
\[
\frac{z(1 + 2A_2z + 3A_3z^2 + \ldots)}{z + A_2z^2 + A_3z^3 + \ldots} = 1 - b + b(1 + c_1z + c_2z^2 + \ldots),
\]
which implies the equality
\[
z + 2A_2z^2 + 3A_3z^3 + \ldots = z + (A_2 + bc_1)z^2 + (A_3 + bc_1A_2 + bc_2)z^3 + \ldots.
\]

Equating the coefficients of both side, we have
\[
A_2 = bc_1, \quad A_3 = \frac{b}{2}(c_2 + bc_1^2),
\] (11)
so that, on account of (10)
\[
a_2 = \frac{b}{2}(c + 2)^\delta c_1, \quad a_3 = \frac{b}{2}(c + 3)^\delta (c_2 + bc_1^2). \] (12)

Taking into account (12) and Lemma 1, we obtain
\[
|a_2| \leq 2|b|\left( \frac{c + 2}{c + 1} \right)^\delta,
\]
and Lemma 2
\[
|a_3| = \left| \frac{b}{2}(c + 3)^\delta (c_2 + bc_1^2) \right| \leq |b|\left( \frac{c + 3}{c + 1} \right)^\delta \max\{1, |1 + 2b|\}.
\]

Moreover, by Lemma 1
\[
|\alpha_3 - \mu a_2^2| = \left| \frac{b}{2}(c + 2)^\delta (c_2 + bc_1^2) - \frac{bc_2}{2}(c + 3)^\delta (c + 2)^{2\delta} \right| \\ = \left| \frac{bc_2}{2}(c + 3)^\delta \right| \leq |b|\left( \frac{c + 3}{c + 1} \right)^\delta,
\]
as asserted.

Now, we consider functional \(|a_3 - \mu a_2^2|\) for complex \( \mu \).

**Theorem 2.** Let \( c, \delta \geq 0; b \in C \setminus \{0\} \). If \( f \in S_{c,\delta}(b) \), then for \( \mu \in C \), we have
\[
|a_3 - \mu a_2^2| \leq |b|\left( \frac{c + 3}{c + 1} \right)^\delta \max\{1, 1 + 2b - 4\mu b\left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^\delta \}.
\]

Moreover for each \( \mu \), there is a function in \( S_{c,\delta}(b) \) such that equality holds.

**Proof.** Taking into account (12) we have
\[
a_3 - \mu a_2^2 = \frac{b}{2}(c + 2)^\delta (c_2 + bc_1^2) - \mu b^2c_1^2\left( \frac{c + 2}{c + 1} \right)^{2\delta} = \frac{b}{2}(c + 2)^\delta (c_2 + \beta c_1^2),
\] (13)
where
\[
\beta = -b + 2\mu b\left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^\delta.
\]

Then, with the aid of Lemma 2, we obtain
\[
|a_3 - \mu a_2^2| \leq |b|\left( \frac{c + 3}{c + 1} \right)^\delta \max\{1, 1 + 2b - 4\mu b\left( \frac{(c + 2)^2}{(c + 1)(c + 3)} \right)^\delta \} \] (14)
as asserted. An examination of the proof shows that equality is attained for the first case when \( c_1 = 0 \) and \( c_2 = 2 \) and the corresponding \( f \in S_{c,\delta}(b) \) is given by
\[
z(I_b^z f(z))' = \frac{1 + (2b - 1)z}{1 - z^2}, \] (15)
and likewise for the second case when \( c_1 = c_2 = 2 \) the corresponding \( f \in S_{c,\delta}(b) \) is given by
\[
z(I_b^z f(z))' = \frac{1 + (2b - 1)z}{1 - z}, \] (16)
respectively.
Taking \( \delta = 0 \) and \( b = 1 \) in Theorem 2, we have:

Corollary 1: [10] If \( f \in S^* \), then for \( \mu \in \mathbb{C} \) we have

\[
|a_3 - \mu a_2^2| \leq \max\{1, |4\mu - 3|\}.
\]

Moreover for each \( \mu \), there is a function in \( S^* \) such that equality holds.

We next consider the case when \( \mu \) and \( b \) are real. Then we have

Theorem 3: Let \( c, \delta \geq 0; b > 0 \). If \( f \in \mathcal{S}_{c, \delta}(b) \), then for \( \mu \in \mathbb{R} \), we have

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{b+c+3}{2c+1} \delta & \text{if } \mu \leq \frac{1}{2} \left( \frac{(c+1)(c+3)}{(c+2)^2} \right) \\
\frac{b+c+3}{2c+1} \left[ 2 - \frac{|c_1|^2}{2} + \frac{|c_2|^2}{2} \left( 1 + 2b - 4\mu b \left( \frac{(c+2)^2}{(c+1)(c+3)} \right) \right) \right] & \text{if } \mu \geq \frac{1}{2} \left( \frac{(c+1)(c+3)}{(c+2)^2} \right)
\end{cases}
\]

Moreover for each \( \mu \), there is a function in \( \mathcal{S}_{c, \delta}(b) \) such that equality holds.

Proof. By (14), we obtain

\[
a_3 - \mu a_2^2 = \frac{b+c+3}{2c+1} \delta \left[ c_2 - \frac{c_2^2}{2} + \frac{c_2^2}{2} \left( 1 + 2b - 4\mu b \left( \frac{(c+2)^2}{(c+1)(c+3)} \right) \right) \right].
\]

First, let \( \mu \leq \frac{1}{2} \left( \frac{(c+1)(c+3)}{(c+2)^2} \right) \), in this case, by (17), Lemma 1 and Lemma 3 give

\[
|a_3 - \mu a_2^2| \leq \frac{b+c+3}{2c+1} \delta \left[ 2 - \frac{|c_1|^2}{2} + \frac{|c_2|^2}{2} \left( 1 + 2b - 4\mu b \left( \frac{(c+2)^2}{(c+1)(c+3)} \right) \right) \right] \leq \frac{b+c+3}{c+1} \delta \left[ 1 + 2b - 4\mu b \left( \frac{(c+2)^2}{(c+1)(c+3)} \right) \right].
\]

Now let \( \mu \geq \frac{1}{2} \left( \frac{(c+1)(c+3)}{(c+2)^2} \right) \). Then, using the above calculations, we get

\[
|a_3 - \mu a_2^2| \leq \frac{b+c+3}{2c+1} \delta.
\]

Finally, if \( \mu \geq \frac{1+b}{2b} \left( \frac{(c+1)(c+3)}{(c+2)^2} \right) \), then we obtain

\[
|a_3 - \mu a_2^2| \leq \frac{b+c+3}{2c+1} \delta \left[ 2 - \frac{|c_1|^2}{2} + \frac{|c_2|^2}{2} \left( -1 - 2b + 4\mu b \left( \frac{(c+2)^2}{(c+1)(c+3)} \right) \right) \right] \leq \frac{b+c+3}{c+1} \delta \left[ 1 - 2b + 4\mu b \left( \frac{(c+2)^2}{(c+1)(c+3)} \right) \right].
\]

Equality is attained for the second case on choosing \( c_1 = 0, c_2 = 2 \) in (15) and in (16) \( c_1 = c_2 = 2; c_1 = 2i, c_2 = -2 \) for the first and third case, respectively. Thus the proof is complete.

Using the relation (9), we easily obtain bounds of coefficients and a solution of the Fekete-Szegő problem in \( \mathcal{C}_{c, \delta} \).

Theorem 4: Let \( c, \delta \geq 0; b \in \mathbb{C}\setminus\{0\} \). If \( f \in \mathcal{C}_{c, \delta}(b) \), then

\[
|a_2| \leq \left| \frac{c+3}{c+1} \right| \delta \max\{1, |1+2b|\},
\]

and

\[
|a_3| \leq \left| \frac{b+c+3}{c+1} \right| \delta \max\{1, |1+2b|\}.
\]

Reasoning in the same line as in proof of Theorem 2 obtain :

Theorem 5: Let \( c, \delta \geq 0; b \in \mathbb{C}\setminus\{0\} \). If \( f \in \mathcal{C}_{c, \delta}(b) \), then for \( \mu \in \mathbb{C} \), we have

\[
|a_3 - \mu a_2^2| \leq \left| \frac{b+c+3}{c+1} \right| \delta \max\left\{1, |1+2b - 3\mu b \left( \frac{(c+2)^2}{(c+1)(c+3)} \right) \right\}.
\]

Moreover for each \( \mu \), there is a function in \( \mathcal{C}_{c, \delta}(b) \) such that equality holds.

By taking \( \delta = 0 \) and \( b = 1 \) in Theorem 5, we have

Corollary 2: [10] If \( f \in \mathcal{C}^* \), then for \( \mu \in \mathbb{C} \) we have

\[
|a_3 - \mu a_2^2| \leq \max\left\{1, |\mu - 1| \right\}.
\]

Moreover for each \( \mu \), there is a function in \( \mathcal{C}^* \) such that equality holds.
REFERENCES


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