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Fekete-Szegö Problem for Subclasses of Analytic Functions Defined by New Integral Operator

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Abstract—The author introduced the integral operator , by using this operator a new subclasses of analytic functions are introduced. For these classes, several Fekete-Szeg¨ type coefficient inequalities are obtained.

Keywords—Integral operator, Fekete-Szeg¨ inequalities, Analytic functions.

I. Introduction and definition

LET $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unite disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.
Also let $\mathcal{S}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are univalent in $\mathbb{U}$.
In [2] Fekete and Szeg¨ proved a noticeable result that the estimate

$$|a_3 - \mu a_2^2| \leq 1 + 2exp\Big(\frac{-2\mu}{1-\mu}\Big)$$

holds for $f \in \mathcal{S}$ and for $0 \leq \mu \leq 1$. This inequality is sharp for each μ. The coefficient functional

$$\phi_\mu(f) = a_3 - \mu a_2^2 = \frac{1}{6}\Big(f'''(0) - \frac{3\mu}{2}(f''(0))^2\Big)$$

on $f \in \mathcal{A}$ represents various geometric quantities as well as in the sense that this behaves well with respect to the rotation, namely

$$\phi_\mu(e^{-i\theta} f(e^{i\theta} z) = e^{2i\theta}\phi_\mu(f), \ \ (\theta \in \mathbb{R}).$$

In fact, other than the simplest case when

$$\phi_0(f) = a_3,$$

we have several important ones. For example,

$$\phi_1(f) = a_3 - a_2^2,$$

represent $S_f(0)/6$, where S_f denotes the Schwarzian derivative

$$S_f(z) = \Big(\frac{f''(z)}{f'(z)}\Big)' - \frac{1}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2.$$

Moreover, the first two non-trivial coefficients of the n-th root transform

$$(f(z^n))^{\frac{1}{n}} == z + c_{n+1}z^{n+1} + c_{2n+1}z^{2n+1} + ...$$

of f with the power series (1), are written by

$$c_{n+1} = \frac{a_2}{n}$$

and

$$c_{2n+1} = \frac{a_3}{n} + \frac{(n-1)a_2^2}{2n^2}$$

so that

$$a_3 - \mu_2^2 = n(c_{2n+1} - \lambda c_{n+1}^2),$$

where

$$\lambda = \mu n + \frac{n-1}{2}$$

Thus, it is quite natural to ask about inequalities for ϕ_μ corresponding to subclasses of $\mathcal{S}$. This is called Fekete-Szeg¨ problem. Actually many authors have considered this problem for typical classes of univalent functions.

Recently, in [1] the author introduced a certain integral operator $\mathcal{I} - c^\delta$ defined by :

$$\mathcal{I}_c^\delta f(z) = \frac{(1+c)^\delta}{\Gamma(\delta)}\int_0^1 t^{c-1}(\log 1/t)^{\delta-1} f(tz)dt, \tag{2}$$

where $c > 0$, $\delta > 1$ and $z \in \mathbb{U}$.

We also note that the operator $\mathcal{I}_c^\delta f(z)$ defined by (1) can be expressed by the series expansion as following:

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$$\mathcal{I}_c^\delta f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+c}{k+c}\right)^\delta a_k z^k, \tag{3}$$

Obviously, we have, for $(\delta, \lambda \geq 0)$

$$\mathcal{I}_c^\delta (I_c^\lambda f(z)) = I_c^{\delta+\lambda} f(z). \tag{4}$$

and

$$\mathcal{I}_c^\delta (zf'(z)) = z(I_c^\delta f(z))'. \tag{5}$$

Moreover, from (3), it follows that

$$z(\mathcal{I}_c^{\delta+1} f(z))' = (c+1)\mathcal{I}_c^\delta f(z) - c\mathcal{I}_c^{\delta+1} f(z) \tag{6}$$

We note that :

- For $c = 0$ and $\delta = n$(n is any integer), the multiplier transformation $\mathcal{I}_0^n f(z) = I^n f(z)$ was studied by Flett [3] and Salagean [4];
- For $c = 0$ and $\delta = -n(n \in \mathbb{N}_0 = \{0, 1, 2, 3...\})$, the differential operator $\mathcal{I}_0^{-n} f(z) = D^n f(z)$ was studied by Salagean [4];
- For $c = 1$ and $\delta = n$(n is any integer), the operator $\mathcal{I}_1^n f(z) = \mathcal{I}^n f(z)$ was studied by Uralegaddi and Somanatha [5];
- For $c = 1$, the multiplier transformation $\mathcal{I}_1^\delta f(z) = \mathcal{I}^\delta f(z)$ was studied by Jung et al. [6];
- For $c = a-1$ $(a > 0)$, the integral operator $\mathcal{I}_{a-1}^\delta f(z) = \mathcal{I}_{a-1}^\delta f(z)$ was studied by Komatu [7];

Using the operator $\mathcal{I}_c^\delta$, we now introduce the following classes:

Definition 1: we say that a function $f \in \mathcal{A}$ is in the class $\mathcal{S}_{c,\delta}(b)$ if

$$\Re\left\{1 + \frac{1}{b}\Big(\frac{z(\mathcal{I}_c^\delta f(z))'}{\mathcal{I}_c^\delta f(z)} - 1\Big)\right\} > 0,$$
$$(c > 0, \delta \geq 0, b \in \mathbb{C}\backslash\{0\},\ z \in \mathbb{U}). \tag{7}$$

Definition 2: we say that a function $f \in \mathcal{A}$ is in the class $\mathcal{C}_{c,\delta}(b)$ if

$$\Re\left\{1 + \frac{1}{b}\frac{z(\mathcal{I}_c^\delta f(z))''}{\mathcal{I}_c^\delta f(z)}\right\} > 0,$$
$$(c > 0, \delta \geq 0, b \in \mathbb{C}\backslash\{0\},\ z \in \mathbb{U}). \tag{8}$$

Note that

$$f \in \mathcal{C}_{c,\delta}(b) \Leftrightarrow zf' \in \mathcal{S}_{c,\delta}(b). \tag{9}$$

In particular, we have starlike and convex function classes, $\mathcal{S}_{c,0}(1) = \mathcal{S}^*$ and $\mathcal{C}_{c,0}(1) = \mathcal{C}$, respectively.

We denote by P a class of the analytic functions in $\mathbb{U}$ with

$$p(0) = 1 \text{ and } \Re\{p(z)\} > 0.$$

To prove our results, we need the following Lemmas considered by Duren [8] Ravichandran et al. [9] .

Lemma 1: [8] Let $p \in P$ with $p(z) = 1 + c_1 z + c_2 z^2 + ...$. Then

$$|c_n| \leq 2, \quad (n \geq 1).$$

Lemma 2: [9] Let $p \in P$ with $p(z) = 1 + c_1 z + c_2 z^2 + ...$. Then for any complex number γ

$$|c_2 - \gamma c_1^2| \leq 2\max\{1, |2\gamma - 1|\},$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

Lemma 3: [8] Let $p \in P$ with $p(z) = 1 + c_1 z + c_2 z^2 + ...$. Then

$$\left|c_2 - \frac{1}{2}\lambda c_1^2\right| \leq 2 + \frac{1}{2}(|\lambda - 1| - 1)|c_1|^2.$$

II. Main results

Theorem 1: Let $c, \delta \geq 0; b \in \mathbb{C}\backslash\{0\}$. If $f \in \mathcal{S}_{c,\delta}(b)$, then

$$|a_2| \leq 2|b|\Big(\frac{c+2}{c+1}\Big)^\delta,$$

$$|a_3| \leq |b|\Big(\frac{c+3}{c+1}\Big)^\delta \max\{1, |1+2b|\},$$

and

$$\left|a_3 - \frac{1}{2}\Big(\frac{(c+1)(c+3)}{(c+2)^2}\Big)^\delta a_2^2\right| \leq |b|\Big(\frac{c+3}{c+1}\Big)^\delta.$$

Proof. Denote

$$\mathcal{I}_c^\delta = z + A_2 z^2 + A3z^3 + ...$$

Then by (3), we can write

$$A_2 = \Big(\frac{c+1}{c+2}\Big)^{\delta} a_2, \;\; A_3 = \Big(\frac{c+1}{c+3}\Big)^{\delta} a_3. \tag{10}$$

by the definition of the class $\mathcal{S}_{c,\delta}(b)$, there exists $p \in P$ such that:

$$1 + \frac{1}{b}\Big(\frac{z(\mathcal{I}_c^{\delta} f(z))'}{\mathcal{I}_c^{\delta} f(z)} - 1\Big) = p(z),$$
$$\frac{z(\mathcal{I}_c^{\delta} f(z))'}{\mathcal{I}_c^{\delta} f(z)} = 1 - b + bp(z),$$

so that

$$\frac{z(1 + 2A_2 z + 3A_3 z^2 + ...)}{z + A_2 z^2 + A_3 z^3 + ...} = 1 - b + b(1 + c_1 z + c_2 z^2 + ...),$$

which implies the equality

$$z + 2A_2 z^2 + 3A_3 z^3 + ...$$
$$= z + (A_2 + bc_1)z^2 + (A_3 + bc_1 A_2 + bc_2)z^3 + ... \,.$$

Equating the coefficients of both side, we have

$$A_2 = bc_1, \;\; A_3 = \frac{b}{2}(c_2 + bc_1^2), \tag{11}$$

so that, on account of (10)

$$a_2 = b\Big(\frac{c+2}{c+1}\Big)^{\delta} c_1, \;\; a_3 = \frac{b}{2}\Big(\frac{c+3}{c+1}\Big)^{\delta}(c_2 + bc_1^2). \tag{12}$$

Taking into account (12) and Lemma 1, we obtain

$$|a_2| \le 2|b|\Big(\frac{c+2}{c+1}\Big)^{\delta},$$

and Lemma 2

$$\begin{aligned} |a_3| &= \Big|\frac{b}{2}\Big(\frac{c+3}{c+1}\Big)^{\delta}(c_2 + bc_1^2)\Big| \\ &\le |b|\Big(\frac{c+3}{c+1}\Big)^{\delta} \max\{1, |1+2b|\}. \end{aligned}$$

Moreover, by Lemma 1

$$\begin{aligned} &\Big|a_3 - \frac{1}{2}\Big(\frac{(c+1)(c+3)}{(c+2)^2}\Big)^{\delta} a_2^2\Big| \\ = &\Big|\frac{b}{2}\Big(\frac{c+3}{c+1}\Big)^{\delta}(c_2 + bc_1^2) - \frac{b^2 c_1^2}{2}\Big(\frac{(c+1)(c+3)}{(c+2)^2}\Big)^{\delta}\Big(\frac{c+2}{c+1}\Big)^{2\delta}\Big| \\ = &\Big|\frac{bc_2}{2}\Big(\frac{c+3}{c+1}\Big)^{\delta}\Big| \\ \le &|b|\Big(\frac{c+3}{c+1}\Big)^{\delta}. \end{aligned}$$

as asserted.

Now, we consider functional $|a_3 - \mu a_2^2|$ for complex μ.

Theorem 2: Let $c, \delta \ge 0; b \in \mathbb{C}\backslash\{0\}$. If $f \in \mathcal{S}_{c,\delta}(b)$, then for $\mu \in \mathbb{C}$, we have

$$\big|a_3 - \mu a_2^2\big| \le |b|\Big(\frac{c+3}{c+1}\Big)^{\delta} \max\Big\{1, \Big|1 + 2b - 4\mu b\Big(\frac{(c+2)^2)}{(c+1)(c+3)}\Big)^{\delta}\Big|\Big\}.$$

Moreover for each μ, there is a function in $\mathcal{S}_{c,\delta}(b)$ such that equality holds.

Proof. Taking into account (12) we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{b}{2}\Big(\frac{c+2}{c+1}\Big)^{\delta}(c_2 + bc_1^2) - \mu b^2 c_1^2\Big(\frac{c+2}{c+1}\Big)^{2\delta} \\ &= \frac{b}{2}\Big(\frac{c+2}{c+1}\Big)^{\delta}(c_2 + \beta c_1^2), \end{aligned} \tag{13}$$

where

$$\beta = -b + 2\mu b\Big(\frac{(c+2)^2)}{(c+1)(c+3)}\Big)^{\delta}.$$

Then, with the aid of Lemma 2, we obtain

$$\begin{aligned} &|a_3 - \mu a_2^2| \\ &\le |b|\Big(\frac{c+3}{c+1}\Big)^{\delta} \max\Big\{1, \Big|1 + 2b - 4\mu b\Big(\frac{(c+2)^2)}{(c+1)(c+3)}\Big)^{\delta}\Big|\Big\}. \end{aligned} \tag{14}$$

as asserted. An examination of the proof shows that equality is attained for the first case when $c_1 = 0$ and $c_2 = 2$ and the corresponding $f \in \mathcal{S}_{c,\delta}(b)$ is given by

$$\frac{z(\mathcal{I}_c^{\delta} f(z))'}{\mathcal{I}_c^{\delta} f(z)} = \frac{1 + (2b-1)z^2}{1 - z^2}, \tag{15}$$

and likewise for the second case when $c_1 = c_2 = 2$ the corresponding $f \in \mathcal{S}_{c,\delta}(b)$ is given by

$$\frac{z(\mathcal{I}_c^{\delta} f(z))'}{\mathcal{I}_c^{\delta} f(z)} = \frac{1 + (2b-1)z}{1 - z}, \tag{16}$$

respectively.

Taking $\delta = 0$ and $b = 1$ in Theorem 2, we have :

Corollary 1: [10] If $f \in \mathcal{S}^*$, then for $\mu \in \mathbb{C}$ we have

$$\left|a_3 - \mu a_2^2\right| \leq \max\{1, |4\mu - 3|\}.$$

Moreover for each μ, there is a function in $\mathcal{S}^*$ such that equality holds.

We next consider the case when μ and b are real. Then we have

Theorem 3: Let $c, \delta \geq 0; b > 0$. If $f \in \mathcal{S}_{c,\delta}(b)$, then for $\mu \in \mathbb{R}$, we have

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} b(\frac{c+3}{c+1})^\delta \left[1 + 2b - 4\mu b\Big(\frac{(c+2)^2}{(c+1)(c+3)}\Big)^\delta\right] \\ \text{if } \mu \leq \frac{1}{2}\Big(\frac{(c+1)(c+3)}{(c+2)^2}\Big)^\delta \\ b\Big(\frac{c+3}{c+1}\Big)^\delta \\ \text{if } \frac{1}{2}\Big(\frac{(c+1)(c+3)}{(c+2)^2}\Big)^\delta \leq \mu \leq \frac{1+b}{2b}\Big(\frac{(c+1)(c+3)}{(c+2)^2}\Big)^\delta \\ b(\frac{c+3}{c+1})^\delta \left[-1 - 2b + 4\mu b\Big(\frac{(c+2)^2}{(c+1)(c+3)}\Big)^\delta\right] \\ \text{if } \mu \geq \frac{1+b}{2b}\Big(\frac{(c+1)(c+3)}{(c+2)^2}\Big)^\delta \end{cases}$$

Moreover for each μ, there is a function in $\mathcal{S}_{c,\delta}(b)$ such that equality holds.

Proof. By (14), we obtain

$$a_3 - \mu a_2^2 = \frac{b}{2}\Big(\frac{c+3}{c+1}\Big)^\delta \left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\Big(1 + 2b - 4\mu b\Big(\frac{(c+2)^2}{(c+1)(c+3)}\Big)\Big)^\delta\right]. \tag{17}$$

First, let $\mu \leq \frac{1}{2}\Big(\frac{(c+1)(c+3)}{(c+2)^2}\Big)^\delta$. in this case, by (17), Lemma 1 and Lemma 3 give

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{b}{2}\Big(\frac{c+3}{c+1}\Big)^\delta \left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\Big(1 + 2b - 4\mu b\Big(\frac{(c+2)^2}{(c+1)(c+3)}\Big)\Big)^\delta\right] \\ &\leq b(\frac{c+3}{c+1})^\delta \left[1 + 2b - 4\mu b\Big(\frac{(c+2)^2}{(c+1)(c+3)}\Big)^\delta\right].\end{aligned}$$

Now let $\frac{1}{2}\Big(\frac{(c+1)(c+3)}{(c+2)^2}\Big)^\delta \leq \mu \leq \frac{1+b}{2b}\Big(\frac{(c+1)(c+3)}{(c+2)^2}\Big)^\delta$. Then, using the above calculations, we get

$$\left|a_3 - \mu a_2^2\right| \leq b\Big(\frac{c+3}{c+1}\Big)^\delta.$$

Finally, if $\mu \geq \frac{1+b}{2b}\Big(\frac{(c+1)(c+3)}{(c+2)^2}\Big)^\delta$, then we obtain

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{b}{2}\Big(\frac{c+3}{c+1}\Big)^\delta \left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\Big(-1 - 2b + 4\mu b\Big(\frac{(c+2)^2}{(c+1)(c+3)}\Big)\Big)^\delta\right] \\ &\leq \frac{b}{2}\Big(\frac{c+3}{c+1}\Big)^\delta \left[2 + \frac{|c_1|^2}{2}\Big(-2 - 2b + 4\mu b\Big(\frac{(c+2)^2}{(c+1)(c+3)}\Big)\Big)^\delta\right] \\ &\leq b(\frac{c+3}{c+1})^\delta \left[-1 - 2b + 4\mu b\Big(\frac{(c+2)^2}{(c+1)(c+3)}\Big)^\delta\right].\end{aligned}$$

Equality is attained for the second case on choosing $c_1 = 0, c_2 = 2$ in (15) and in (16) $c_1 = c_2 = 2; c_1 = 2i, c_2 = -2$ for the firs and third case, respectively. Thus the proof is complete.

Using the relation (9), we easily obtain bounds of coefficients and a solution of the Fekete-Szegö problem in $\mathcal{C}_{c,\delta}$.

Theorem 4: Let $c, \delta \geq 0; b \in \mathbb{C}\backslash\{0\}$. If $f \in \mathcal{C}_{c,\delta}(b)$, then

$$|a_2| \leq |b|\Big(\frac{c+2}{c+1}\Big)^\delta,$$

$$|a_3| \leq \frac{|b|}{3}\Big(\frac{c+3}{c+1}\Big)^\delta \max\{1, |1+2b|\},$$

and

$$\left|a_3 - \frac{2}{3}\Big(\frac{(c+1)(c+3)}{(c+2)^2}\Big)^\delta a_2^2\right| \leq \frac{|b|}{3}\Big(\frac{c+3}{c+1}\Big)^\delta.$$

Reasoning in the same line as in proof of Theorem 2 obtain :

Theorem 5: Let $c, \delta \geq 0; b \in \mathbb{C}\backslash\{0\}$. If $f \in \mathcal{C}_{c,\delta}(b)$, then for $\mu \in \mathbb{C}$, we have

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| \;\; \leq \;\; & \frac{|b|}{3}\Big(\frac{c+3}{c+1}\Big)^\delta \\ & \max\left\{1, \left|1 + 2b - 3\mu b\Big(\frac{(c+2)^2)}{(c+1)(c+3)}\Big)^\delta\right|\right\}.\end{aligned}$$

Moreover for each μ, there is a function in $\mathcal{C}_{c,\delta}(b)$ such that equality holds.

By taking $\delta = 0$ and $b = 1$ in Theorem 5, we have

Corollary 2: [10] If $f \in \mathcal{C}^*$, then for $\mu \in \mathbb{C}$ we have

$$\left|a_3 - \mu a_2^2\right| \leq \max\{\frac{1}{3}, |\mu - 1|\}.$$

Moreover for each μ, there is a function in $\mathcal{C}^*$ such that equality holds.

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