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Hopf Bifurcation for a New Chaotic System

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Abstract—In this paper, a three dimensional autonomous chaotic system is considered. The existence of Hopf bifurcation is investigated by choosing the appropriate bifurcation parameter. Furthermore, formulas for determining the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions are derived with the help of normal form theory. Finally, a numerical example is given.

Keywords—Chaotic system, Hopf bifurcation, normal form theory.

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I. Introduction

IN 1963, Lorenz found the first canonical chaotic system in [1] as follows,

$$\begin{cases} \dot{x} = a(y-x), \\ \dot{y} = cx - xz - y, \\ \dot{z} = xy - bz. \end{cases} \tag{1}$$

The system has been extensively studied in the fields of chaos theory, dynamical systems as well as chaos control and synchronization. Later, an even simpler chaotic system was constructed in [2]:

$$\begin{cases} \dot{x} = -(x+y), \\ \dot{y} = x + ay, \\ \dot{z} = zx - cz + b. \end{cases} \tag{2}$$

From then on, some other chaotic systems were established, such as Chen system [3], Lü system [4], Liu system [5], Qi system [6], T system [7] and so on. Basic dynamical properties of these systems were studied by means of theoretical analysis, numerical simulation, Lyapunov exponent spectrum, bifurcation diagrams and Poincaré section diagrams. The chaotic systems have great potential applications in secure communications.

In this paper, we mainly consider a three dimensional autonomous chaotic system proposed by Wang et al. [8–9] in the form

$$\begin{cases} \dot{x} = a(x-y), \\ \dot{y} = -cy + xz, \\ \dot{z} = -bz + dxy, \end{cases} \tag{3}$$

where $(x, y, z) \in \mathbf{R}^3$ and $a, b, c, d \in \mathbf{R}$. It has a chaotic attractor as shown in Fig.1 when $a = 20$, $b = 2$, $c = 28$ and $d = 1$. For system (3), stability of equilibria and heteroclinic orbit of Shil'nikov type have been investigated. However, the relationship between the Hopf bifurcation and the system parameters has not been clarified yet.

The aim of this paper is to study the Hopf bifurcation from equilibrium by taking one coefficient as bifurcation parameter. By applying normal from theory and center manifold theorem, the direction of Hopf bifurcation and the stability of bifurcating periodic solutions are presented. Finally, a numerical example is given to support the analytic results.

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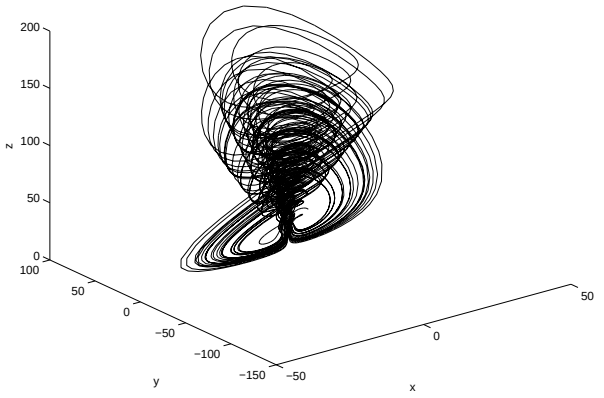


Fig. 1. Chaotic attractor of system (3) with $a = 20$, $b = 2$, $c = 28$ and $d = 1$.

II. Local stability and existence of Hopf bifurcation

By simple analysis, it is easy to obtain that if $bcd < 0$, then system (3) only has one equilibrium $O(0, 0, 0)$; if $bcd > 0$, then system (3) has three equilibria $O(0, 0, 0)$, $E_1(x_0, y_0, z_0)$ and $E_2(-x_0, -y_0, z_0)$, where $x_0 = y_0 = \sqrt{bcd}/d$, $z_0 = c$.

Lemma 2.1. For system (3), we have the following results:
(i) if $a > 0$, $b < 0$ and $c < 0$, then $O(0, 0, 0)$ is asymptotically stable;
(ii) if $a < 0$ or $b > 0$ or $c > 0$, then $O(0, 0, 0)$ is unstable;
(iii) if $b + c - a > 0$, $abc > 0$ and $ab(a - b - 3c) > 0$, then E_1 and E_2 are asymptotically stable.

Proof. (i) and (ii) are obvious, we mainly consider the third result. Let $x_1 = x - x_0$, $y_1 = y - y_0$, $z_1 = z - z_0$, we can shift the equilibrium to the origin:

$$\begin{cases} \dot{x}_1 = a(x_1 - y_1), \\ \dot{y}_1 = -cy_1 + x_0 z_1 + z_0 x_1 + x_1 z_1, \\ \dot{z}_1 = -bz_1 + dx_0 y_1 + dy_0 x_1 + dx_1 y_1. \end{cases} \tag{4}$$

The characteristic equation of system (4) is

$$f(\lambda) = \lambda^3 + (b + c - a)\lambda - ab\lambda + 2abc = 0. \tag{5}$$

Let $A = b+c-a$, $B = -ab$ and $C = 2abc$. By Routh–Hurwitz criteria, the roots of (5) have strictly negative real parts if and only if $A > 0$, $C > 0$ and $AB - C > 0$. Then we have

$$b + c - a > 0, \quad abc > 0, \quad ab(a - b - 3c) > 0.$$

This completes the proof.

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Assume $a < 0 < b$, condition (iii) in Lemma 2.1 can be simplified as

$$\frac{a-b}{3} < c < 0.$$

Hence, $A > 0$, $B > 0$, $C > 0$, and we have $f(\lambda) > 0$ for any $\lambda > 0$. There is an instable equilibrium only if there are a pair of complex conjugate roots for (5). Let these two roots be $\lambda_{1,2} = \pm \mathrm{i}\omega$, then we have

$$\lambda_1 + \lambda_2 + \lambda_3 = a - b - c.$$

Therefore, $\lambda_3 = a - b - c$, which is on the margin of stability for system (3). Then we have

$$f(\lambda_3) = -ab(a - b - 3c),$$

and

$$c = c_0 = \frac{a-b}{3}.$$

Thus, Hopf bifurcation may occur at E_1 and E_2. Next, we will prove that the positive equilibrium E_1 will lose its stability when $c = c_0$.

Theorem 2.2. Assume $a < 0 < b$ and $c < 0$, $d < 0$, when c passes through the critical value $c_0 = \frac{a-b}{3}$, system (3) undergoes a Hopf bifurcation at the equilibrium E_1.

Proof. If $c = c_0$, then equation (5) is equivalent to

$$(\lambda^2 - ab)(\lambda + \frac{2(b-a)}{3}) = 0.$$

Therefore, the characteristic equation has a pair of purely imaginary roots $\lambda_{1,2} = \pm \mathrm{i}\sqrt{-ab}$ and a negative real root $\lambda_3 = \frac{2(a-b)}{3}$.

Differentiating both sides of equation (5) with respect to c, we obtain

$$\frac{\mathrm{d}\lambda}{\mathrm{d}c} = -\frac{\lambda^2 + 2ab}{3\lambda^2 + 2\lambda(b + c - a) - ab},$$

and

$$\left.\frac{\mathrm{dRe}\lambda}{\mathrm{d}c}\right|_{c=c_0} = -\frac{54a^2b^2}{36a^2b^2 - 16ab(a-b)^2} < 0,$$

$$\left.\frac{\mathrm{dIm}\lambda}{\mathrm{d}c}\right|_{c=c_0} = \frac{36ab(b-a)\sqrt{-ab}}{36a^2b^2 - 16ab(a-b)^2} < 0.$$

According to Hopf bifurcation theorem in [10], we can conclude that c_0 is the critical value. The equilibrium E_1 is stable when $c > c_0$ and there exist periodic solutions when $c < c_0$. The conclusions follows.

III. Properties of Hopf bifurcation

In this section, we shall derive the explicit formulae determining the direction, stability, and period of these periodic solutions bifurcating from E_1 at c_0, by using techniques from normal form theory and center manifold theorem [10].

Let the eigenvectors corresponding to the eigenvalues $\lambda_1 = \mathrm{i}\sqrt{-ab}$ and $\lambda_3 = 2c_0$ be $u_1 + \mathrm{i}u_2$ and u_3. By direct calculations, we get

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ -\frac{\omega}{a} \\ \frac{d\sqrt{-a}(a-c)}{a\sqrt{cd}} \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ \frac{a-2c}{a} \\ \frac{ad}{2bc\sqrt{bcd}} \end{pmatrix}.$$

Define

$$P = (u_1, -u_2, u_3) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & \frac{\omega}{a} & \frac{a-2c}{a} \\ 0 & -\frac{d(a-c)}{\sqrt{-acd}} & \frac{ad}{2bc\sqrt{bcd}} \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = P \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}.$$

Then

$$\begin{cases} \dot{x}_2 = -\sqrt{-ab}y_2 + F_1(x_2, y_2, z_2), \\ \dot{y}_2 = \sqrt{-ab}x_2 + F_2(x_2, y_2, z_2), \\ \dot{z}_2 = (a - b - c)z_2 + F_3(x_2, y_2, z_2), \end{cases} \tag{6}$$

where

$$F_1(x_2, y_2, z_2) = -d(x_2 + z_2)\left(x_2 + \frac{\sqrt{-ab}}{a}y_2 + \frac{a-2c}{a}z_2\right),$$

$$F_2(x_2, y_2, z_2) = d(x_2 + z_2)\left[-\frac{4bc^2\sqrt{bcd}}{a\sqrt{-ab}}x_2 - \left(\frac{4bc^2\sqrt{bcd}}{a^2} + \frac{ad(a-c)}{\sqrt{-acd}}\right)y_2 + \left(\frac{a^2d}{2bc\sqrt{bcd}} - \frac{4bc^2\sqrt{bcd}}{a\sqrt{-ab}}\right)z_2\right],$$

$F_3(x_2, y_2, z_2) = -F_1(x_2, y_2, z_2)$.

According to the procedures proposed by Hassard et al. [10], we can get

$$\begin{aligned} g_{11} &= \frac{1}{4}\left(\frac{\partial^2 F_1}{\partial x_2^2} + \frac{\partial^2 F_1}{\partial y_2^2} + \mathrm{i}\left(\frac{\partial^2 F_2}{\partial x_2^2} + \frac{\partial^2 F_2}{\partial y_2^2}\right)\right) \\ &= -\frac{d}{4}\left(1 + \mathrm{i}\frac{4bc^2\sqrt{bcd}}{a\sqrt{-ab}}\right), \end{aligned}$$

$$\begin{aligned} g_{02} &= \frac{1}{4}\left(\frac{\partial^2 F_1}{\partial x_2^2} - \frac{\partial^2 F_1}{\partial y_2^2} - 2\frac{\partial^2 F_2}{\partial x_2 \partial y_2} + \mathrm{i}\left(\frac{\partial^2 F_2}{\partial x_2^2} - \frac{\partial^2 F_2}{\partial y_2^2} + 2\frac{\partial^2 F_1}{\partial x_2 \partial y_2}\right)\right) \\ &= -\frac{d}{4}\left(1 - 2\left(\frac{4bc^2\sqrt{bcd}}{a^2} + \frac{ad(a-c)}{\sqrt{-acd}}\right)\right) \\ &\quad -\mathrm{i}\frac{d}{4}\left(\frac{4bc^2\sqrt{bcd}}{a\sqrt{-ab}} + 2\frac{\sqrt{-ab}}{a}\right), \end{aligned}$$

$$\begin{aligned} g_{20} &= \frac{1}{4}\left(\frac{\partial^2 F_1}{\partial x_2^2} - \frac{\partial^2 F_1}{\partial y_2^2} + 2\frac{\partial^2 F_2}{\partial x_2 \partial y_2} + \mathrm{i}\left(\frac{\partial^2 F_2}{\partial x_2^2} - \frac{\partial^2 F_2}{\partial y_2^2} - 2\frac{\partial^2 F_1}{\partial x_2 \partial y_2}\right)\right) \\ &= -\frac{d}{4}\left(1 + 2\left(\frac{4bc^2\sqrt{bcd}}{a^2} - \frac{ad(a-c)}{\sqrt{-acd}}\right)\right) \\ &\quad -\mathrm{i}\frac{d}{4}\left(\frac{4bc^2\sqrt{bcd}}{a\sqrt{-ab}} - 2\frac{\sqrt{-ab}}{a}\right), \end{aligned}$$

$$\begin{aligned} G_{21} &= \frac{1}{8}\left(\frac{\partial^3 F_1}{\partial x_2^3}+\frac{\partial^3 F_1}{\partial x_2\partial y_2^2}+\frac{\partial^3 F_2}{\partial x_2^2\partial y_2}+\frac{\partial^3 F_2}{\partial y_2^3}+\mathrm{i}\left(\frac{\partial^3 F_2}{\partial x_2^3}\right.\right.\\ &\quad \left.\left.+\frac{\partial^3 F_2}{\partial x_2\partial y_2^2}-\frac{\partial^3 F_1}{\partial x_2^2\partial y_2}-\frac{\partial^3 F_1}{\partial y_2^3}\right)\right)\\ &= 0. \end{aligned}$$

From the dimension $n = 3 > 2$, we calculate the following,

$$h_{11} = \frac{1}{4}\left(\frac{\partial^2 F_3}{\partial x_2^2}+\frac{\partial^2 F_3}{\partial y_2^2}\right) = \frac{d}{4},$$

$$h_{20} = \frac{1}{4}\left(\frac{\partial^2 F_3}{\partial x_2^2}-\frac{\partial^2 F_3}{\partial y_2^2}-2\mathrm{i}\frac{\partial^2 F_3}{\partial x_2\partial y_2}\right) = \frac{1}{4}\left(d-2\mathrm{i}\frac{d\sqrt{-ab}}{a}\right).$$

By solving the linear equations

$$\lambda_3\omega_{11} = -h_{11},$$

$$(\lambda_3 - 2\mathrm{i}\sqrt{-ab})\omega_{20} = -h_{20},$$

we obtain

$$\omega_{11} = -\frac{d}{4(a-b-c)},$$

$$\omega_{20} = -\frac{d(a-5b-c)+\mathrm{i}\left[2d\sqrt{-ab}-2(a-b-c)\frac{d\sqrt{-ab}}{a}\right]}{4(a-b-c)^2-16ab}.$$

Furthermore, we have

$$\begin{aligned} G_{110} &= \frac{1}{2}\left(\frac{\partial^2 F_1}{\partial x_2\partial z_2}+\frac{\partial^2 F_1}{\partial y_2\partial z_2}+\mathrm{i}\left(\frac{\partial^2 F_2}{\partial x_2\partial z_2}-\frac{\partial^2 F_2}{\partial y_2\partial z_2}\right)\right)\\ &= -\frac{d}{2}\left(2-\frac{2c}{a}+\frac{\sqrt{-ab}}{a}\right)+\mathrm{i}\frac{d}{2}\left(\frac{a^2 d}{2bc\sqrt{bcd}}\right.\\ &\quad \left.-\frac{8bc^2\sqrt{bcd}}{a\sqrt{-ab}}+\frac{4bc^2\sqrt{bcd}}{a^2}+\frac{ad(a-c)}{\sqrt{-acd}}\right), \end{aligned}$$

$$\begin{aligned} G_{101} &= \frac{1}{2}\left(\frac{\partial^2 F_1}{\partial x_2\partial z_2}-\frac{\partial^2 F_1}{\partial y_2\partial z_2}+\mathrm{i}\left(\frac{\partial^2 F_2}{\partial x_2\partial z_2}+\frac{\partial^2 F_2}{\partial y_2\partial z_2}\right)\right)\\ &= -\frac{d}{2}\left(2-\frac{2c}{a}-\frac{\sqrt{-ab}}{a}\right)+\mathrm{i}\frac{d}{2}\left(\frac{a^2 d}{2bc\sqrt{bcd}}\right.\\ &\quad \left.-\frac{8bc^2\sqrt{bcd}}{a\sqrt{-ab}}-\frac{4bc^2\sqrt{bcd}}{a^2}-\frac{ad(a-c)}{\sqrt{-acd}}\right). \end{aligned}$$

So we can compute the following quantities:

$$g_{21} = G_{21} + (2G_{110}\omega_{11} + G_{101}\omega_{20}),$$

$$c_1(0) = \frac{1}{2\omega}\left(g_{20}g_{11}-2|g_{11}|^2-\frac{1}{3}|g_{02}|^2\right)+\frac{1}{2}g_{21},$$

$$\mu_2 = -\frac{\mathrm{Re}(c_1(0))}{\mathrm{Re}(\lambda'(c_0))},$$

$$\beta_2 = 2\mathrm{Re}(c_1(0)),$$

$$\tau_2 = -\frac{\mathrm{Im}(c_1(0))+\mu_2\mathrm{Im}(\lambda'(c_0))}{\omega}.$$

It is well known that μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0(\mu_2 < 0)$, then the Hopf bifurcation is subcritical (supercritical) and the bifurcating periodic solutions exist for $c > c_0(c < c_0)$; β_2 determines the stability of bifurcating periodic solutions: the bifurcating periodic solutions on the center manifold are stable (unstable) if $\beta_2 < 0(\beta_2 > 0)$; and τ_2 determines the periods of the bifurcating periodic solutions: the periods increase (decreases) if $\tau_2 > 0(\tau_2 < 0)$.

From the proof of Theorem 2.2, we know that $\mathrm{Re}(\lambda'(c_0)) < 0$, therefore we have the following result.

Theorem 3.2. The direction of the Hopf bifurcation of (3) at E_1 is supercritical (subcritical) and the bifurcating periodic solutions on the center manifold are stable (unstable) if $\mathrm{Re}(c_1(0)) < 0(\mathrm{Re}(c_1(0)) > 0)$.

IV. Numerical example

Next, we shall give a numerical example of system (3). Let $a = -5$, $b = 5$ and $d = -3$, we can compute the Hopf bifurcation value $c_0 = -\frac{10}{3}$. The equilibrium is stable when $c = -2 > c_0$ and unstable when $c = -4 < c_0$, as shown in Figs.2 and 3, respectively. From the formulas in previous section, we have $c_1(0) = -553.81375 - 1060.87537\mathrm{i}$ when $c = -4$. Thus, the periodic solutions bifurcating from positive equilibrium is supercritical and stable.

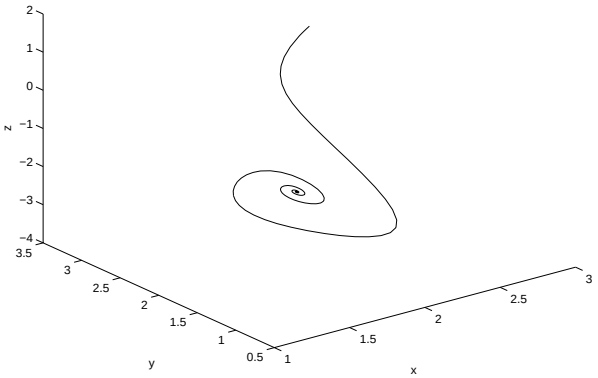


Fig. 2. Phase diagram of system (3) with $a = -5$, $b = 5$, $c = -2$ and $d = -3$.

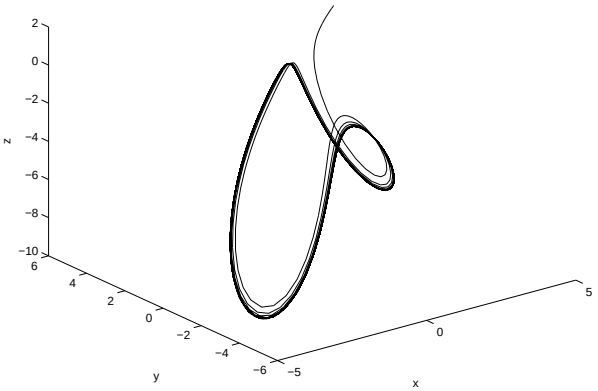


Fig. 3. Phase diagram of system (3) with $a = -5$, $b = 5$, $c = -4$ and $d = -3$.

V. Conclusions

In this paper, a three dimensional autonomous chaotic system has been studied. By choosing an appropriate bifurcation parameter, we prove that Hopf bifurcation occurs when

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the bifurcation parameter passes through the critical value. The direction of the Hopf bifurcation and stability of the bifurcating periodic solutions are analyzed in detail.

Apparently there are more interesting problems about this chaotic system in terms of complexity, control and synchronization, which deserve further investigation.

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