Skolem Sequences and Erdosian Labellings of m Paths with 2 and 3 Vertices

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Abstract—Assume that we have m identical graphs where the graphs consists of paths with k vertices where k is a positive integer. In this paper, we discuss certain labelling of the m graphs called c-Erdösian for some positive integers c. We regard labellings of the vertices of the graphs by positive integers, which induce the edge labels for the paths as the sum of the two incident vertex labels. They have the property that each vertex label and edge label appears only once in the set of positive integers $\{c, \ldots, c + 6m - 1\}$. Here, we show how to construct certain c-Erdösian of m paths with 2 and 3 vertices by using Skolem sequences.

Keywords—c-Erdösian, Skolem sequences, magic labelling

I. INTRODUCTION

GRAPH labellings are assignment of integers to the vertices or edges, or both, subject to certain conditions. In 1963, Sedláček [7] introduced magic labellings for graphs. A connected graph is said to be semi-magic if there is a labelling of the edges with integers such that for each vertex v the sum of the labels of all edges incident with v is the same for all v. A semi-magic labelling where the edges are labelled with distinct positive integers is called a magic labelling. In 1970, Kotzig and Rosa [4] introduced magic labellings of a graph $G(V, E)$ as a bijection f from $V \cup E$ to $\{1, 2, \ldots, |V \cup E|\}$ such that for all edges xy, $f(x) + f(y) + f(xy)$ is constant and this type of graph labelling is called edge-magic total labelling. In 1999, MacDougall, Miller, Slamin, and Wallis [5] introduced the notion of a vertex-magic total labelling. For a graph $G(V, E)$, an injective mapping f from $V \cup E$ to the set $\{1, 2, \ldots, |V| + |E|\}$ is a vertex-magic total labelling if there is a constant k, called the magic constant, such that for every vertex v, $f(v) + \sum f(vu) = k$ where the sum is over all vertices u adjacent to v.

Let $m \cdot P_k = (V_m, E_m)$ be the finite (disconnected) graph with vertex set V of size $|V| = km$ and edge set E_m of size $|E_m| = km$, consisting of m disjoint paths. When $k = 2$, $m \cdot P_2$ represents m disjoint paths with 2 vertices and when $k = 3$, $m \cdot P_3$ represents m disjoint paths with 3 vertices. A *total labelling* of the graph $m \cdot P_k$ is a positive integer valued function $f : V_m \cup E_m \to N$. A labelling is said to be *magic* if its range consists of the integers $\{1, 2, \ldots, (2k-1)m\}$ and it is said to be *c-magic* if its range consists of the integers $\{c, c+1, \ldots, c+(2k-1)m-1\}$, for any positive integer $c > 0$.

We say that f is a *c-Erdösian triangle labelling* if it is $c-$magic and if it has the following property: For any edge

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$xy \in E_m$, with $x, y \in V_m$ we have

$$f(x) + f(y) = f(xy).$$

For convenience, we say that $m \cdot P_k$ is *c-Erdösian* if it satisfies the conditions above for any positive integer k.

Let $m \cdot P_k$ consists of m disjoint paths with k vertices. Let $D_i^2 = \{a_i, b_i, a_i + b_i\}$, $i = 1, \ldots, m$ be their vertex and edge labels for each path with 2 vertices. By using the similar terminology, we have $D_i^3 = \{a_i, b_i, c_i, a_i + b_i, b_i + c_i\}$, $i = 1, \ldots, m$ be the vertex and edge labels for each path with 3 vertices. In other word, the system $\{D_1^k, \ldots, D_m^k\}$ (for $k = 2, 3$) is called *c-Erdösian* if its range consists of the integers $\{c, c+1, \ldots, c+(2k-1)m-1\}$, for any positive integer $c > 0$. For convenience, the elements of D_i^k can be represented in the following form of a 2-by-k array, where the top row indicates the vertex labels and the bottom row shows the edge labels as shown below for the case $k = 2$ and 3 respectively:

$$\begin{array}{cc} a_i & b_i \\ a_i + b_i & \end{array}$$

$$\begin{array}{ccc} a_i & b_i & c_i \\ a_i + b_i & & b_i + c_i \end{array}$$

In this paper, we only consider graph which consists of m paths with 2 or 3 vertices. The following is an example of the $2 \cdot P_3$ which is 3-Erdösian.

$$\begin{array}{cccccc} 9 & 3 & 5 & 7 & 4 & 6 \\ 12 & & 8, & 11 & & 10. \end{array}$$

Skolem sequences and their generalizations have been used widely in the construction of combinatorial design. A *Skolem sequence of order n* is a sequence $S = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying the conditions

(S1) for every $k \in \{1, 2, \ldots, n\}$ there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$;

(S2) if $s_i = s_j = k$ with $i < j$, then $j - i = k$.

A *hooked Skolem sequence of order n* is a sequence $S = (s_1, s_2, \ldots, s_{2n+1})$ of $2n+1$ integers satisfying the conditions (S1) and (S2) above and

(S3) $s_{2n} = 0$.

To construct our c-Erdösian of m paths, we will use the Langford sequence if $c > 1$. A *Langford sequence* of order n and defect d, $n > d$, is a sequence $L = (l_1, l_2, \ldots, l_{2n})$ of $2n$ integers satisfying the conditions

(L1) for every $k \in \{d, d+1, \ldots, d+n-1\}$ there exist exactly two elements $l_i, l_j \in L$ such that $l_i = l_j = k$, and

(L2) if $l_i = l_j = k$ with $i < j$, then $j - i = k$.

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The *hooked Langford* sequences of order n and defect d is a sequence $L = (l_1, l_2, \ldots, l_{2n+1})$ of $2n+1$ integers satisfying conditions (L1) and (L2) above and

(L3) $l_{2n} = 0$.

Clearly, a (hooked) Langford sequence with defect 1 is a (hooked) Skolem sequence. It is well-known that a Skolem sequence of order n exists if and only if $n \equiv 0,1$ (mod 4) [9] and a hooked Skolem sequence of order n exists if and only if $n \equiv 2,3$ (mod 4) [6]. The following theorem gives necessary and sufficient conditions for the existence of Langford sequences.

Theorem 1: [8] A Langford sequence of order n and defect d exists if and only if

(i) $n \geq 2d - 1$, and
(ii) $n \equiv 0,1$ (mod 4) and d is odd, or $n \equiv 2,3$ (mod 4) and d is even.

A hooked Langford sequence of order n and defect d exists if and only if

(i) $n(n - 2d + 1) + 2 \geq 0$, and
(ii) $n \equiv 2,3$ (mod 4) and d is odd, or $n \equiv 1,2$ (mod 4) and d is even.

II. SKOLEM SEQUENCES AND ERDOSIAN LABELLINGS OF m PATHS P_2

We begin by considering the paths with two vertices.

Proposition 1: The path P_2 is $1-$Erdösian but it is not c-Erdösian for $c \geq 2$.

Proof. Let x_1, x_2 is an enumeration of the vertices of P_2. If X be the sum of the vertex labels, then $X \geq c + (c+1) = 2c+1$. Note that the total sum of all the labels is $2X = c + (c+1) + (c+2) = 3c+3$, and it follows that $X = \frac{1}{2}(3c+3)$. Therefore $\frac{1}{2}(3c+3) \geq 2c+1$, and hence $c \leq 1$. $\quad\square$

For the case $c = 1$, there is only a $1-$Erdösian labelling for P_2 as follow:

$$\begin{matrix} 1 & 2 \\ 3 & \end{matrix}$$

We first look at the following necessary condition:

Proposition 2: If $m \cdot P_2$ is c-Erdösian, then $c \leq \frac{m+1}{2}$.

Proof. Let D_i^2 be the set of vertex and edge labels for i-th path with 2 vertices where $D_i^2 = \{a_i, b_i, a_i + b_i\}$, $i = 1, 2, \ldots, m$. Note that $D_1^2 \cup \cdots \cup D_m^2 = \{c, c+1, \ldots, c+3m-1\}$. Let TS be the sum of the vertices and BS be the sum of edges. Then, $TS = a_i + b_i$ and $BS = (a_i + b_i) = TS$ for all $i = 1, \ldots, m$. Note that

$$TS \geq c + (c+1) + \cdots + [c + (2m-1)] = m(2c + 2m - 1)$$

and

$$BS \leq (c + 2m) + \cdots + [c + (3m-1)] = \frac{m}{2}(2c + 5m - 1).$$

Then $m(2c + 2m - 1) \leq \frac{m}{2}(2c + 5m - 1)$ and it follows that $c \leq \frac{m+1}{2}$. $\quad\square$

By referring to [9], [1] and [2], we recall that a skolem sequence of order n exists if and only if $n \equiv 0,1$ (mod 4). When $n = 1$, we take $(1,1)$. When $n = 4$, take $(1,1,3,4,2,3,2,4)$ and if we rewrite in ordered pairs, we have $(1,2), (5,7), (3,6), (4,8)$. When $n = 5$, take $(2,4,2,3,5,4,3,1,1,5)$. When $n > 5$, we use the construction of order paired as follows:

$$n = 4s : \begin{cases} (4s - r + 1, 8s - r + 1), & r = 1, \ldots, 2s; \\ (r, 4s - r - 1), & r = 1, \ldots, s-2; \\ (s + r + 1, 3s - r), & r = 1, \ldots, s-2; \\ (s - 1, 3s), (s, s+1), (2s, 4s-1), & (2s+1, 6s) \end{cases}$$

$$n = 4s+1 : \begin{cases} (4s + r - 1, 8s - r + 3), & r = 1, \ldots, 2s; \\ (r, 4s - r + 1), & r = 1, \ldots, s; \\ (s + r + 2, 3s - r + 1), & r = 1, \ldots, s-2; \\ (s + 1, s + 2), (2s + 1, 6s + 2), & (2s + 2, 4s + 1) \end{cases}$$

The following result is clear by the construction of Skolem sequences above.

Theorem 2: There exists an 1-Erdösian of $m \cdot P_2$ when $m \equiv 0, 1$ (mod 4).

Example 1: When $n = 4$, we have the following skolem sequences in ordered pair:

$$(1, 2), (5, 7), (3, 6), (4, 8).$$

By adding n to all the value in the ordered pairs above and writing the difference of the ordered pair in the first position of triples, we have the set of triples

$$\{(1, 5, 6), (2, 9, 11), (3, 7, 10), (4, 8, 12)\}.$$

The first two values in the triples will be the vertex labels for each path P_2 and hence we get an 1-Erdösian for $4 \cdot P_2$ easily by using Skolem sequences. $\quad\square$

We can use the similar argument to construct a c-Erdösian of $m \cdot P_2$ if there exists a Langford sequence of order m and defect c. For example, given a Langford sequence of order 5 and defect 3, $L = (7, 5, 3, 6, 4, 3, 5, 7, 4, 6)$, we obtain the triples as follows:

$$\{(3, 10, 13), (4, 12, 16), (5, 9, 14), (6, 11, 17), (7, 8, 15)\}.$$

Hence, the first two values in the triples will be the vertex labels and the last values will be the edge label for each path P_2.

We note that the c-Erdösian of $m \cdot P_2$ exists if there exists a Langford sequence of order m and defect c. From the Theorem 1, we have the following:

Theorem 3: There exists a c-Erdösian of $m \cdot P_2$ if

(i) $m \geq 2c - 1$, and
(ii) $m \equiv 0,1$ (mod 4) and c is odd, or $m \equiv 2,3$ (mod 4) and c is even.

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III. ERDÖSIAN LABELLINGS OF m PATHS P_3

In this section, we shall consider the paths with three vertices.

Proposition 3: If $m \cdot P_3$ is c-Erdösian, then $c \leq \frac{2m+1}{2}$.

Proof. Let D_i^3 be the set of vertex and edge labels for i-th path with 3 vertices where $D_i^3 = \{a_i, b_i, c_i, a_i + b_i, b_i + c_i\}$, $i = 1, 2, \ldots, m$. Note that $D_1^3 \cup \cdots \cup D_m^3 = \{c, c+1, \ldots, c+5m-1\}$ and sum of the first two vertex labels $a_i + b_i \geq c + (c+1) + \cdots + [c+(2m-1)] = m(2c+2m-1)$. Similarly, we have the other sum of the two vertex labels is $b_i + c_i \geq m(2c + 2m - 1)$ and it follows that

$$a_i + 2b_i + c_i \geq 2m(2c + 2m - 1).$$

The sum of the edge labels

$$(a_i + b_i) + (b_i + c_i) \leq (c+3m) + \cdots + (c+5m-1) = m(2c+8m-1)$$

It follows that

$$2m(2c+2m-1) \leq (c+3m) + \cdots + (c+5m-1) = m(2c+8m-1)$$

Therefore $c \leq \frac{2m+1}{2}$. $\square$

For the case $m = 1$, P_3 is 1-Erdösian as given below:

$$
\begin{array}{ccc}
2 & 1 & 4 \\
3 & & 5.
\end{array}
$$

Skolem sequences can also be used to construct $m \cdot P_3$ if $m \equiv 0, 1 \pmod 4$. However the way of constructing the paths with 3 vertices is not that direct as given in the $m \cdot P_2$. We use the skolem sequence and the property of c-Erdösian triangle labelling for m triangles for construction. We continue this section by introducing the definition of the c-Erdösian triangle labelling for m-triangles:

Let $GT_m = (V_m, E_m)$ be the finite (disconnected) graph with vertex set V of size $|V| = 3m$ and edge set E_m of size $|E_m| = 3m$, consisting of m disjoint triangles K_3, that is we let $GT_m = m \cdot K_3$. A *total labelling* of the graph GT_m is a positive integer valued function $f : V_m \cup E_m \to N$. In [3], we say that f is a *c-Erdösian triangle labelling* if it is c−magic and if it has the following property: For any edge $xy \in E_m$, with $x, y \in V_m$ we have

$$f(x) + f(y) = f(xy).$$

For convenience, we say that GT_m is *c-Erdösian* if it satisfies the conditions above.

Proposition 4: If the sum of the vertex labels over any one triangle is a constant, then

$$a_i + b_i + c_i = 2c + 6m - 1 \text{ for all } i = 1, 2, \ldots, m. \quad (1)$$

Proof. The constant in (1) is obtained by summing over all labels,

$$
\begin{aligned}
c + (c+1) + \cdots + (c+6m-1) &= 6mc + 3m(6m-1) \\
&= 3m(2c + 6m - 1)
\end{aligned}
$$

and then dividing this expression by $3m$, since there are m triangles and each triangle has total sum of labels $3(a_i + b_i + c_i)$. $\square$

However, not all the GT_m is c-Erdösian with constant vertex labels over the m triangles. The following is an example which the sum of the vertex labels over each triangle which is all distinct.

$$
\begin{array}{cccccccccccc}
1 & 2 & 15 & & 4 & 5 & 23 & & 6 & 7 & 18 \\
3 & 16 & 17, & & 9 & 27 & 28, & & 13 & 24 & 25,
\end{array}
$$

$$
\begin{array}{cccccc}
8 & 12 & 14 & & 10 & 11 & 19 \\
20 & 22 & 26, & & 21 & 29 & 30.
\end{array}
$$

The results below are straightforward by the Proposition 4.

Proposition 5: There exists an 1-Erdösian of $m \cdot P_3$ when $m \equiv 0, 1 \pmod 4$.

Theorem 4: For any $1 \leq c \leq \frac{m+1}{2}$, there exists an c-Erdösian of $m \cdot P_3$ when

(i) $m \equiv 0, 1 \pmod 4$ and c is odd, or

(ii) $m \equiv 2, 3 \pmod 4$ and c is even.

Example 2: Given a Langford sequence in triples for $m = 3$ and $c = 2$ as follow: $\{(2, 7, 9), (3, 5, 8), (4, 6, 10)\}$.

The first two integers in each triple of are the vertex labels of a triangle and the third vertex label of the triangle can be obtained by using Proposition 4. So the 2-Erdösian of GT_3 is clear from the following 2-by-3 arrays:

$$
\begin{array}{cccccccccc}
2 & 7 & 12 & & 3 & 5 & 13 & & 4 & 6 & 11 \\
9 & 14 & 19, & & 8 & 16 & 18, & & 10 & 15 & 17.
\end{array}
$$

By deleting the largest edge label in each triangle, we have the following 2-Erdösian of $3 \cdot P_3$:

$$
\begin{array}{ccccccccc}
7 & 2 & 12 & & 5 & 3 & 13 & & 6 & 4 & 11 \\
9 & & 14, & & 8 & & 16, & & 10 & & 15.
\end{array} \quad \square
$$

Hooked Langford sequences can also be used to construct $m \cdot P_3$ as shown in the next example.

Example 3: From the hooked Langford sequences for the case $m = 5$ and $c = 2$, we can rewrite in triples as

$$\{(2, 7, 9), (3, 12, 15), (4, 10, 14), (5, 8, 13), (6, 11, 17)\}.$$

Since $c_i = 2(2) + 6(5) - 1 - a_i - b_i = 33 - a_i - b_i$, we have the following 2-Erdösian arrays of 5 triangles:

$$
\begin{array}{cccccccccc}
2 & 7 & 24 & & 3 & 12 & 18 & & 4 & 10 & 19 \\
9 & 26 & 31, & & 15 & 21 & 30, & & 14 & 23 & 29,
\end{array}
$$

$$
\begin{array}{cccccc}
5 & 8 & 20 & & 6 & 11 & 16 \\
13 & 25 & 28, & & 17 & 22 & 27.
\end{array}
$$

Hence, the 2-Erdösian of $5 \cdot P_3$ are listed below:

$$
\begin{array}{cccccccccc}
7 & 2 & 24 & & 12 & 3 & 18 & & 10 & 4 & 19 \\
9 & & 26, & & 15 & & 21, & & 14 & & 23,
\end{array}
$$

$$
\begin{array}{ccccccc}
8 & 5 & 20 & & 11 & 6 & 16 \\
13 & & 25, & & 17 & & 22. \square
\end{array}
$$

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By using the hooked Langford sequence, we can construct $m \cdot P_3$ which is c-Erdösian under certain conditions given by Theorem 1 as follows:

Theorem 5: A c-Erdösian of $m \cdot P_3$ exists if

(i) $m(m - 2c + 1) + 2 \geq 0$, and

(ii) $m \equiv 2,3 \pmod 4$ and c is odd, or $m \equiv 1,2 \pmod 4$ and c is even.

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