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Modeling and stability analysis of delayed game network

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Abstract—This paper aims to establish a delayed dynamical relationship between payoffs of players in a zero-sum game. By introducing Markovian chain and time delay in the network model, a delayed game network model with sector bounds and slope bounds restriction nonlinear function is first proposed. As a result, a direct dynamical relationship between payoffs of players in a zero-sum game can be illustrated through a delayed singular system. Combined with Finsler's Lemma and Lyapunov stable theory, a sufficient condition guaranteeing the unique existence and stability of zero-sum game's Nash equilibrium is derived. One numerical example is presented to illustrate the validity of the main result.

Keywords—Game networks, zero-sum game, delayed singular system, nonlinear perturbation, time delay.

I. Introduction

Game Theory is the study of optimal decision making under competition when one individual's decisions affect the outcome of a situation for another individuals involved. It is a mathematical method for analyzing calculated circumstances. Since the Von Neumann's original proof used Brouwer's fixed-point theorem on continuous mappings into compact convex sets [1], game theory attracted many scholars' interest. In the research of game theory, much effort has been devoted to developing all kinds of game theories. Up to now, many excellent papers and monographs have been published [2-11]. General speaking, Game Theory can be broadly classified into four main sub-categories of study: classical game theory, combinatorial game theory, dynamic game theory, and other game theory. Today, game theory applies to a wide range of class relations, and has developed into an umbrella term for the logical side of science, to include both human and non-humans. As a result, it has been widely recognized as an important tool in many fields such as economics, biology, engineering, political science, control theory, computer science, and philosophy.

Recently, some famous game such as prisoner's dilemma and snowdrift game have been applied to investigate the cooperation game [12-14]. However, as pointed out in [15] that, the most concerned issue is the payoffs for individuals in a game, it is desirable to construct a direct dynamical relationship among the payoffs. Hence, in this paper, we will focus our interest on the differential dynamics between payoffs of players in a zero-sum mixed strategy game. In order to describe the zero-sum property in a game, singular model [16-18] is a good selection. On the basis of the network topology among the payoffs for players in a game, Xiong , Daniel, and Cao [15] established a singular game network model with Markovian chain to describe the zero-sum mixed strategy game, and analyzed the dynamic behavior.

It worth pointing out that the model established in [15] only considered the current payoff impacts among the players and the external issues. In fact, the payoffs of every individuals in a game not only relate to the other's current payoffs and external issues, but also related to the other players' previous payoffs. It is a result of all kinds of comprehensive factors. This idea motivates this study. In order to describe the relationship among players' current payoffs, previous payoffs and external disturbance factors, we first analyzed the delayed network topology among these factors for a zero-sum game, and established a delayed singular game network model with Markovian jumping parameters. Then, combined with Finsler's Lemma and Lyapunov stable theory, a sufficient condition guaranteeing the unique existence and stability of zero-sum game's Nash equilibrium is derived. At last, one numerical simulation example is presented to illustrate the validity of the given result.

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II. Mathematic Modeling of Delayed Game Network

Assume that there are n persons in a game, and a pure action profile of player i is s_i, the set of pure strategies be S_i, namely $s_i \in S_i$. Let $S = \prod_i S_i$ be the set of all pure action profiles for all players. Let $s \in S$ be pure strategy for all players, when n persons perform this pure strategy to play a game, every player's payoff is clearly related to the other's payoffs. Moreover, the payoff of a player $i(i = 1, 2, \cdots, n)$ may be affected by other player's previous payoffs and some external issues such as economical power, reputation, resources and so on. Fig.1 is given to show the relationship between the payoff of player j and those of other players in performing a strategy s, where x_i denotes the payoff of player i; Weighting parameters a_{ji} and b_{ji} $(i = 1, 2, \cdots, n)$ reflect the current and previous impacts in performing a strategy s from player i to player j respectively. $f_j(x_j(t))$ represents the external uncertain influence which is caused by player $j's$ own condition, and c_{ji} is the weighting parameter. In order to obtain a Nash equilibrium, a mixed strategy σ is needed, and assume σ of all players is probability measure over the pure strategy set S. Let Σ denote the set of mixed strategies of players, and S is known, total number is m, a mixed strategy Σ^i of player i is a probability measure over pure strategy s_i^j, where s_i^j is implemented with probability p_j satisfying $\sum_{j=1}^{m} p_j = 1,\ p_j \geq 0\ (j = 1, 2, \cdots, m)$. Obversely, the probability p_j of pure strategy s_i^j is related to the probabilities of other pure strategy in a mixed strategy. In order to describe this mixed strategy property, a appropriate way is Markovian

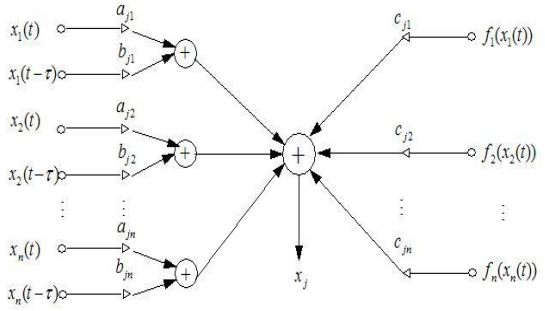


Fig. 1. Delayed Network Topology of Zero-sum Game

chain. Additional, in order to describe the zero-sum property in a zero-sum game, singular model can also be introduced to simulate the differential dynamics among the payoffs of all the players in a zero-sum mixed strategy game. On the basis of these analysis, we can establish a delayed zero-sum differential game model with n persons as follows:

$$\dot{x}(t) = \widetilde{A}(r(t))x(t) + \widetilde{B}_d(r(t))x(t-\tau) + \widetilde{C}_g(r(t))g(x(t)), \quad (1)$$

subject to

$$x_1(t) + x_2(t) + \cdots + x_n(t) = 0. \quad (2)$$

Here, $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ is a vector of payoffs for all players; $\{r(t), t \geq 0\}$ is a condition-time Markovian process with right-continuous trajectories and taking values in a finite set $\Psi = \{1, 2, \cdots, m\}$ with transition probability matrix $\Pi = (\pi_{ij})$ $(i, j \in \Psi)$ given by

$$p\{r(t+h) = j | r(t) = i\} = \begin{cases} \pi_{ij}h + o(h), & i \neq j, \\ 1 + \pi_{ii}h + o(h), & i = j, \end{cases} \quad (3)$$

where $h > 0$ and $\lim\limits_{h\to 0} \frac{o(h)}{h} = 0$, $\pi_{ij} \geq 0$ is transition rate from i to j if $i \neq j$, and $\pi_{ii} = -\sum_{j=1, j\neq i}^{m} \pi_{ij}$. $\widetilde{A}(r(t)), \widetilde{B}_d(r(t))$, and $\widetilde{C}_g(r(t))$ are known real constant matrices with appropriate dimensions for each $r(t) \in \Psi$; τ is time delay satisfying $\tau > 0$; $g(x(t))$ denotes the nonlinear perturbation.

Notice that $\dot{x}_1(t) + \dot{x}_2(t) + \cdots + \dot{x}_n(t) = 0$, model (1) and (2) can be rewritten as small

$$E\dot{y}(t) = A(r(t))y(t) + B(r(t))y(t-\tau) + C(r(t))f(y(t)), \quad (4)$$

where

$$y(t) = [x^T(t), 0]^T, E = \begin{bmatrix} I_n & 0 \\ \mathbf{1}_n & 0 \end{bmatrix}, f(y(t)) = \begin{bmatrix} g(x(t)) \\ 0 \end{bmatrix},$$

$$A(r(t)) = \begin{bmatrix} \widetilde{A}(r(t)) & 0 \\ 0 & 1 \end{bmatrix}, B(r(t)) = \begin{bmatrix} \widetilde{B_d}(r(t)) & 0 \\ 0 & 1 \end{bmatrix},$$

$$C(r(t)) = \begin{bmatrix} \widetilde{C_g}(r(t)) & 0 \\ 0 & 1 \end{bmatrix}, and\ \mathbf{1}_n = [\underbrace{1, 1, \cdots, 1}_{n}].$$

Remark 2.1: As mentioned in [8], for a differential equation of a game, there exist a close relation between the rest point and the Nash equilibrium as follows:
1) If y is a Nash equilibrium, then it is a rest point;
2) If the rest point y is stable, then it is a Nash equilibrium;
This conclusion implies that the research about the existence of Nash equilibrium in a differential game can be transmitted to the research on the stability problem of the rest point.

Based on the above result obtained in [8], in what follows, we will focus on the asymptotically stable analysis for the rest point of model (4). Before deriving the main results, the following assumption, definition, and lemmas are needed.

Assumption: $f(0) = 0$ and $f(y(t))$ is restricted by the sector bounds $[l_i^-, l_i^+]$, i.e

$$l_i^- \leq \frac{f_i(y_i(t))}{y_i(t)} \leq l_i^+, \ i = 1, 2, \cdots, n.$$

Remark 2.2: As assumption $f(0) = 0$, one can easily obtain that $0_n = (0, 0, \cdots, 0)^T$ is the rest point of system (4), this implies that if we obtain the stochastic asymptotic stability of 0_n, then 0_n is a Nash equilibrium of system (4), this suggests us to analyze the stability of the rest point in system (4).

Notice that the nonlinear function $f_i(\cdot)$ can be written as a convex combination of the sector bounds as follows:

$$f_i(y_i(t)) = (\lambda_i(y_i(t))l_i^- + (1 - \lambda_i(y_i(t)))l_i^+)y_i(t),$$

where $\lambda_i(y_i) = \frac{f_i(y_i(t)) - l_i^- y_i(t)}{(l_i^+ - l_i^-)y_i(t)}$ satisfying $0 \leq \lambda_i(y_i) \leq 1$. Namely, $f_i(y(t)) = \Lambda_i(y_i(t))y_i(t)$, where $\Lambda_i(y_i(t))$ is an element of a convex hull $Co\{l_i^-, l_i^+\}$. Let us define

$$\Lambda = \text{diag}\{\Lambda_1(y_1(t)), \Lambda_2(y_2(t)), \cdots, \Lambda_n(y_n(t))\},$$

$$\Delta_1 = \text{diag}\{l_1^-, l_2^-, \cdots, l_n^-\}, \ \Delta_2 = \text{diag}\{l_1^+, l_2^+, \cdots, l_n^+\}.$$

Then, nonlinearity $f(y(t))$ can be expressed as $f(y(t)) = \Lambda y(t)$. Set $\Omega = \{\Lambda \mid \Lambda \in Co\{\Delta_1, \Delta_2\}\}$.

Definition 2.1: The system (4) is said to be stochastically asymptotically stable, if for any $y_0 \in R^n$ and $r_0 \in \Psi$, there exists a positive scalar $M(y_0, r_0)$ such that

$$\lim_{t\to+\infty} \mathscr{E}\{\int_0^t \| y(t, y_0, r_0) \|^2 \, dt | y_0, r_0\} < M(y_0, r_0),$$

where $y(t, y_0, r_0)$ denotes the solution of system (4) at time t under the initial conditions y_0 and r_0.

Lemma 2.1: [19] Let a matrix F, a symmetric matrix $Q = Q^T$ and a compact subset of real matrices h be given. The following statements are equivalent:
(1) For each $H \in h$, $\xi^T Q \xi < 0$, for all $\xi \neq 0$ such that $HF\xi \neq 0$.
(2) There exists $\Theta = \Theta^T$ such that $Q + F^T \Theta F < 0$, $\Psi_h^T \Theta \Psi_h \geq 0$, for all $H \in h$, where Ψ_h is a matrix belong to a null space of H.

Lemma 2.2: [20] For any positive definite symmetric constant matrix Q and scalar $\tau > 0$, such that the following integrations are well defined, then

$$-\int_{-\tau}^{0}\int_{t+\theta}^{t} y^T(s)Qy(s)dsd\theta$$

$$\leq -\frac{1}{\tau^2}(\int_{-\tau}^{0}\int_{t+\theta}^{t} y(s)dsd\theta)^T Q(\int_{-\tau}^{0}\int_{t+\theta}^{t} y(s)dsd\theta).$$

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III. Stability Analysis of the Rest Point

In this section, we attempt to establish a practically computable stochastic stability criterion for system (4). By constructing a Lyapunov functional including tripe-integral item, we obtain the following stability result.

Theorem 3.1: For given scalar $\tau > 0$, positive definite diagonal matrices $L^- = diag(l_1^-, l_2^-, \cdots, l_n^-)$, $L^+ = diag(l_1^+, l_2^+, \cdots, l_n^+)$, singular system (4) is stochastically asymptotically stable if there exist positive definite diagonal matrices $D = diag\{d_1, d_2, \cdots, d_n\}$, $\Lambda_1 = diag\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, $\Lambda_2 = diag\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, $\Gamma = diag\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, symmetric positive definite matrices Q_1, Q_2, Q_3, symmetric matrix Θ, and arbitrary matrices F_1, N_1, M_1, M_2, P_i $(i \in \Psi)$ of appropriate dimensions such that for every $i \in \Psi$,

$$(1)\ E^T P_i = P_i^T E \geq 0,$$

$$(2)\ \begin{bmatrix} I \\ \omega \end{bmatrix}^T \Theta \begin{bmatrix} I \\ \omega \end{bmatrix} \geq 0, \forall \omega \in \Omega,$$

$$(3)\ \Xi = \begin{bmatrix} \Xi_1 + \Xi_2^T \Theta \Xi_2 & F^T \\ * & -\frac{\tau^2}{2} E^T Q_3 E \\ * & * \end{bmatrix} < 0,$$

where $\Xi_1 = (\Xi_{ij})$ $i, j = 1, 2, \cdots, 7$, and

$$\Xi_{11} = \tau^2 \Lambda_2 (L^+ - L^-) + 2\tau Q_2 + Q_1 - 2L^- \Gamma L^+ + P_i^T A_i + A_i^T P_i + \sum_{j=1}^{m} \pi_{ij} E^T P_j,$$

$$\Xi_{12} = P_i^T B_i,\ \Xi_{13} = L^+ D - L^- \Lambda_1 + A_i^T M_2^T,$$

$$\Xi_{14} = \Gamma(L^+ - L^-) + A_i^T M_1^T + P_i^T C_i,$$

$$\Xi_{22} = -Q_1,\ \Xi_{23} = -B_i^T M_2^T,\ \Xi_{24} = B_i^T M_1^T,$$

$$\Xi_{33} = \frac{\tau^2}{2} E^T Q_3 E - M_2 E - E^T M_2^T,$$

$$\Xi_{34} = \Lambda_1^T + D^T - E^T M_1^T + M_2 C_i,$$

$$\Xi_{44} = -\Gamma - \Gamma^T + M_1 C_i, F = [0, 0, 0, 0, 0, 0, F_1],$$

$$\Xi_{55} = \frac{2}{\tau} Q_2,\ \Xi_{66} = -\frac{2}{\tau^2} \Lambda_2 (L^+ - L^-),\ \Xi_{77} = -F_1 - F_1^T,$$

$$\Xi_2 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ \tau N_1 & 0 & 0 & 0 & -N_1 & 0 & -N_1 \end{bmatrix}.$$

Proof: Choose a new class of Lyapunov functional candidate as follows:

$$V(y(t), r(t)) = V_1(y(t)) + V_2(y(t)) + V_3(y(t)) + V_4(y(t)),$$

where

$$V_1(y(t)) = y^T(t) E^T P(r(t)) y(t) + 2\sum_{i=1}^{n} \{ \int_0^{y_i(t)} \lambda_i (f_i(s) - l_i^- s) ds + \int_0^{y_i(t)} d_i (l_i^+ s - f_i(s)) ds + \int_{t-\tau}^{t} y^T(s) Q_1 y(s) ds \},$$

$$V_2(y(t)) = 2\sum_{i=1}^{n} \{ \int_{-\tau}^{0} \int_{\theta}^{0} \int_{t+\mu}^{t} \alpha_i y_i(s) [f_i(y_i(s)) - l_i^- y_i(s)] ds d\mu d\theta \} + 2\sum_{i=1}^{n} \{ \int_{-\tau}^{0} \int_{\theta}^{0} \int_{t+\mu}^{t} \alpha_i y_i(s) [l_i^+ y_i(s) - f_i(y_i(s))] ds d\mu d\theta \} + 2 \int_{-\tau}^{0} \int_{t+\theta}^{t} y^T(s) Q_2 y(s) ds d\theta,$$

$$V_3(y(t)) = \int_{-\tau}^{0} \int_{\theta}^{0} \int_{t+\mu}^{t} \dot{y}^T(s) E^T Q_3 E \dot{y}(s) ds d\mu d\theta.$$

Let $\mathcal{A}$ be the weak infinitesimal generator of random process $\{y(t), r(t)\}$. Then for each $r(t) = i$ $(i \in \Psi)$, we have

$$\mathcal{A}V(y(t), r(t) = i) = \lim_{\Delta \to 0} \frac{1}{\Delta} [\mathscr{E}\{V(y(t+\Delta), r(t+\Delta)) | y(t), r(t) = i\} - V(y(t), r(t) = i)].$$

Let $P(r(t) | r(t) = i) = P_i, (i \in \Psi)$,

$$V_{11}(y(t), r(t) = i) = y^T(t) E^T P(r(t)) y(t),$$

$$V_{12}(y(t), r(t) = i) = 2\sum_{i=1}^{n} \{ \int_0^{y_i(t)} \lambda_i (f_i(s) - l_i^- s) ds.$$

$$\begin{aligned} &\mathcal{A}V_{11}(y(t), r(t) = i) \\ &= \lim_{\Delta \to 0} \frac{1}{\Delta} [\mathscr{E}\{V_{11}(y(t+\Delta), r(t+\Delta)) | y(t), r(t) = i\} - V_{11}(y(t), r(t) = i)] \\ &= \lim_{\Delta \to 0} \frac{\sum_{j=1, j \neq i}^{m} y^T(t+\Delta) E^T P_j y(t+\Delta) [\pi_{ij} \Delta + o(\Delta)]}{\Delta} \\ &\quad \frac{y^T(t+\Delta) E^T P_i y(t+\Delta) [1 + \pi_{ii} \Delta + o(\Delta)] - y^T(t) E^T P_i y(t)}{\Delta} \\ &= \lim_{\Delta \to 0} [\sum_{j=1}^{m} y^T(t+\Delta) \pi_{ij} E^T P_j y(t+\Delta) \\ &\quad + \frac{y^T(t+\Delta) E^T P_i y(t+\Delta) - y^T(t) E^T P_i y(t)}{\Delta}] \\ &= 2y^T(t) P_i^T E \dot{y}(t) + y^T(t) (\sum_{j=1}^{m} \pi_{ij} E^T P_j) y(t). \end{aligned}$$

$$\begin{aligned} &\mathcal{A}V_{12}(y(t), r(t) = i) \\ &= \lim_{\Delta \to 0} \frac{2 \sum_{j=1, j \neq i}^{m} \{\sum_{i=1}^{n} (\int_0^{y_i(t+\Delta)} \lambda_i (f_i(s) - l_i^- s) ds)\} [\pi_{ij} \Delta + o(\Delta)]}{\Delta} \\ &\quad \frac{+2(\sum_{i=1}^{n} \int_0^{y_i(t+\Delta)} \lambda_i (f_i(s) - l_i^- s) ds)[1 + \pi_{ii} \Delta + o(\Delta)]}{\Delta} \\ &\quad \frac{-2(\sum_{i=1}^{n} \int_0^{y_i(t)} \lambda_i (f_i(s) - l_i^- s) ds)}{\Delta} \\ &= \lim_{\Delta \to 0} \frac{2\{\sum_{i=1}^{n} (\int_0^{y_i(t+\Delta)} \lambda_i (f_i(s) - l_i^- s) ds)\}}{\Delta} \\ &\quad \frac{\times \{\sum_{j=1, j \neq i}^{m} \pi_{ij} \Delta + 1 + \pi_{ii} \Delta + 2o(\Delta)\}}{\Delta} \\ &\quad \frac{-2(\sum_{i=1}^{n} \int_0^{y_i(t)} \lambda_i (f_i(s) - l_i^- s) ds)}{\Delta} \end{aligned}$$

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$$
\begin{aligned}
&= \lim_{\Delta\to 0}\frac{2\{\sum_{i=1}^{n}(\int_0^{y_i(t+\Delta)}\lambda_i(f_i(s)-l_i^- s)ds)\}}{\Delta}\\
&\quad\frac{-2(\sum_{i=1}^{n}\int_0^{y_i(t)}\lambda_i(f_i(s)-l_i^- s)ds)}{\Delta}\\
&= \lim_{\Delta\to 0}\frac{2\sum_{i=1}^{n}(\int_{y_i(t)}^{y_i(t+\Delta)}\lambda_i(f_i(s)-l_i^- s)ds)}{\Delta}\\
&= \lim_{\Delta\to 0}\frac{2\sum_{i=1}^{n}\lambda_i(f_i(\zeta)-l_i^-\zeta)(y_i(t+\Delta)-y_i(t))}{\Delta}\\
&\qquad(\zeta\in[\min(y_i(t),y_i(t+\Delta)),\max(y_i(t),y_i(t+\Delta))])\\
&= \lim_{\Delta\to 0}2\sum_{i=1}^{n}\lambda_i(f_i(\zeta)-l_i^-\zeta)\frac{y_i(t+\Delta)-y_i(t)}{\Delta}\\
&= 2\sum_{i=1}^{n}\lambda_i(f_i(t)-l_i^- t)\dot{y}_i(t).
\end{aligned}
$$

Similarly, the other weak infinitesimal generators of the rest items can also be computed, which are omitted here, and the weak infinitesimal generator of random process $\{y(t), r(t)\}$ along the trajectory of system (4) is given as $\mathcal{A}V(y(t), r(t)=i) = \mathcal{A}V_1(y(t), r(t)=i) + \mathcal{A}V_2(y(t), r(t)=i) + \mathcal{A}V_3(y(t), r(t)=i)$, where

$$
\begin{aligned}
&\mathcal{A}V_1(y(t), r(t)=i)\\
&= \lim_{\Delta\to 0}\frac{1}{\Delta}[\mathscr{E}\{V_1(y(t+\Delta), r(t+\Delta))|y(t), r(t)=i\}\\
&- V_1(y(t), r(t)=i)]\\
&= 2y^T(t)P_i^T E\dot{y}(t) + y^T(t)(\sum_{j=1}^{m}\pi_{ij}E^T P_j)y(t)\\
&+ 2[f^T(y(t)) - y^T(t)L^-]\Lambda_1\dot{y}(t)\\
&+ 2[y^T(t)L^+ - f^T(y(t))]D\dot{y}(t)\\
&+ y^T(t)Q_1 y(t) - y^T(t-\tau)Q_1 y(t-\tau),
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
&\mathcal{A}V_2(y(t), r(t)=i)\\
&= 2\sum_{i=1}^{n}\{\int_{-\tau}^{0}\int_{\theta}^{0}\alpha_i y_i(t)[f_i(y_i(t)) - l_i^- y_i(t)]d\mu d\theta\\
&- \int_{-\tau}^{0}\int_{\theta}^{0}\alpha_i y_i(t+\mu)[f_i(y_i(t+\mu)) - l_i^- y_i(t+\mu)]d\mu d\theta\}\\
&+ 2\sum_{i=1}^{n}\{\int_{-\tau}^{0}\int_{\theta}^{0}\alpha_i y_i(t)[l_i^+ y_i(t) - f_i(y_i(t))]d\mu d\theta\\
&- \int_{-\tau}^{0}\int_{\theta}^{0}\alpha_i y_i(t+\mu)[l_i^+ y_i(t+\mu) - f_i(y_i(t+\mu))]d\mu d\theta\}\\
&+ 2\tau y^T(t)Q_2 y(t) - 2\int_{t-\tau}^{t} y^T(s)Q_2 y(s)ds\\
&= 2\sum_{i=1}^{n}\{\alpha_i y_i(t)[f_i(y_i(t)) - l_i^- y_i(t)]\frac{\tau^2}{2}\\
&- \int_{-\tau}^{0}\int_{t+\theta}^{t}\alpha_i y_i(s)[f_i(y_i(s)) - l_i^- y_i(s)]dsd\theta\}\\
&+ 2\sum_{i=1}^{n}\{\alpha_i y_i(t)[l_i^+ y_i(t) - f_i(y_i(t))]\frac{\tau^2}{2}\\
&- \int_{-\tau}^{0}\int_{t+\theta}^{t}\alpha_i y_i(s)[l_i^+ y_i(s) - f_i(y_i(s))]dsd\theta\}\\
&+ 2\tau y^T(t)Q_2 y(t) - 2\int_{t-\tau}^{t} y^T(s)Q_2 y(s)ds\\
&= 2\sum_{i=1}^{n}\{\frac{\alpha_i\tau^2}{2}y_i(t)[l_i^+ - l_i^-]y_i(t)\\
&- 2\int_{t-\tau}^{t} y^T(s)Q_2 y(s)ds + 2\tau y^T(t)Q_2 y(t)\\
&- \int_{-\tau}^{0}\int_{t+\theta}^{t}\alpha_i y_i(s)[l_i^+ - l_i^-]y_i(s)dsd\theta.
\end{aligned}
\tag{6}
$$

Notice that, $\sum_{i=1}^{n}\alpha_i\tau^2 y_i(t)[l_i^+ - l_i^-]y_i(t) = \tau^2 y^T(t)\ \Lambda_2[L^+ - L^-]y(t)$. From Lemma 2.2, we have

$$
\begin{aligned}
&2\sum_{i=1}^{n}\{-\int_{-\tau}^{0}\int_{t+\theta}^{t}\alpha_i y_i(s)[l_i^+ - l_i^-]y_i(s)dsd\theta\}\\
&= -2\int_{-\tau}^{0}\int_{t+\theta}^{t} y^T(s)\Lambda_2[L^+ - L^-]y(s)dsd\theta\\
&\le -\frac{2}{\tau^2}(\int_{-\tau}^{0}\int_{t+\theta}^{t} y(s)ds)^T\Lambda_2[L^+ - L^-](\int_{-\tau}^{0}\int_{t+\theta}^{t} y(s)dsd\theta).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\mathcal{A}V_2(y(t), r(t)=i)\\
&\le y^T(t)[\tau^2\Lambda_2(L^+ - L^-) + 2\tau Q_2]y(t)\\
&- (\int_{-\tau}^{0}\int_{t+\theta}^{t} y(s)dsd\theta)^T\frac{2}{\tau^2}\Lambda_2(L^+ - L^-)\int_{-\tau}^{0}\int_{t+\theta}^{t} y(s)dsd\theta\\
&- \frac{2}{\tau}(\int_{t-\tau}^{t} y(s)ds)^T Q_2(\int_{t-\tau}^{t} y(s)ds),\\
&\mathcal{A}V_3(y(t), r(t)=i)\\
&= \int_{-\tau}^{0}\int_{\theta}^{0}[\dot{y}^T E^T(t)Q_3 E\dot{y}(t)\\
&- \dot{y}^T(t+\mu)E^T Q_3 E\dot{y}(t+\mu)]d\mu d\theta\\
&= \frac{\tau^2}{2}\dot{y}^T(t)E^T Q_3 E\dot{y}(t) - \int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{y}^T(s)E^T Q_3 E\dot{y}(s)dsd\theta.
\end{aligned}
\tag{7}
$$

Since $Q_3 > 0$, then $E^T Q_3 E > 0$, as the processing method used in [21], for arbitrary matrix $F_1, F = [0,0,0,0,0,0,F_1]$ of appropriate dimensions, we have

$$
\begin{aligned}
&- \int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{y}^T(s)E^T Q_3 E\dot{y}(s)dsd\theta\\
&\le -2\int_{-\tau}^{0}[\int_{t+\theta}^{t}\dot{y}^T(s)dsF\xi_i(t)]d\theta\\
&- \int_{-\tau}^{0}\theta\xi_i^T(t)F^T(E^T Q_3 E)^{-1}F\xi_i(t)d\theta,\\
&= -2\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{y}^T(s)dsd\theta F\xi_i(t)\\
&+ \frac{\tau^2}{2}\xi_i^T(t)F^T(E^T Q_3 E)^{-1}F\xi_i(t).
\end{aligned}
\tag{8}
$$

where $\xi_i^T(t) = [y^T(t),\ y(t-\tau),\ \dot{y}^T(t),\ f^T(y(t)),\ \int_{t-\tau}^{t} y^T(s)ds,\ \int_{-\tau}^{0}\int_{t+\theta}^{t} y^T(s)dsd\theta,\ \int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{y}^T(s)dsd\theta]^T$.

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Combined with inequalities (7)-(8), we have

$$\begin{aligned}\mathcal{A}V_3(y(t),r(t)=i) \le & \frac{\tau^2}{2}\dot{y}^T(t)E^TQ_3E\dot{y}(t)\\ & + \frac{\tau^2}{2}\xi_i^T(t)F^T(E^TQ_3E)^{-1}F\xi_i(t) \\ & - 2\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{y}^T(s)dsd\theta F\xi_i(t).\end{aligned} \tag{9}$$

Furthermore, for positive definite diagonal matrix Γ, arbitrary matrices M_1, M_2 of appropriate dimensions, we have

$$-2f^T(y(t))\Gamma f(y(t))+2y^T(t)\Gamma(L^++L^-)f(y(t))-2y^T(t)L^-\Gamma L^+y(t)\ge 0,$$

$$2f^T(y(t))M_1[-E\dot{y}(t)+A_iy(t)+B_iy(t-\tau)+C_if(y(t))]=0,$$
$$2\dot{y}^T(t)M_2[-E\dot{y}(t)+A_iy(t)+B_iy(t-\tau)+C_if(y(t))]=0,$$

Using convex properties of the nonlinear perturbation function $f(y(t))$, the following constraint is satisfied:

$$f(y(t))=\Lambda y(t),$$

For any matrices N_1 of appropriate dimensions, by using Newton-Leibnitz formula, we can obtain the following zero equation:

$$\Lambda N_1[\tau y(t)-\int_{t-\tau}^{t}y(s)ds-\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{y}(s)dsd\theta]=0.$$

Namely,

$$\left[\begin{array}{ccccccc}\Lambda & 0 & 0 & -I & 0 & 0 & 0\\ \tau\Lambda N_1 & 0 & 0 & 0 & \Lambda N_1 & 0 & \Lambda N_1\end{array}\right]\xi_i(t)=0.$$

This can be represented as

$$[\omega,-I]\Xi_2\xi_i(t)=0,\omega\in\Omega.$$

If there exists a symmetric matrix Θ satisfying

$$\Xi_1+\Xi_2^T\Theta\Xi_2<0,\left[\begin{array}{c}I\\ \omega\end{array}\right]^T\Theta\left[\begin{array}{c}I\\ \omega\end{array}\right]\ge 0,\forall\omega\in\Omega.$$

Then, by lemma 2.1 and Schur complement [22], if the conditions in Theorem 3.1 hold, we can obtain

$$\mathcal{A}V(y(t),r(t)=i)\le-\epsilon\xi_i^T(t)\Xi\xi_i(t),$$

This implies that there must exist a positive scalar $\epsilon>0$ such that

$$\mathcal{A}V(y(t),r(t)=i)\le-\epsilon\parallel\xi_i(t)\parallel^2\le-\epsilon\parallel y_i(t,y_0,r_0)\parallel^2.$$

By Dynkin's formula, one has

$$\begin{aligned}&\mathscr{E}\{V(y(t),r(t))|y_0,r_0\}-V(y_0,r_0)\\ &\le\mathscr{E}\{\int_0^t\mathcal{A}V(y(s),r(s))ds|y_0,r_0\}\\ &\le-\epsilon\mathscr{E}\{\int_0^t\parallel y(s),r(s))\parallel^2 ds|y_0,r_0\},\end{aligned} \tag{10}$$

this implies

$$\lim_{t\to+\infty}\mathscr{E}\{\int_0^t\parallel y(t,y_0,r_0)\parallel^2 dt|y_0,r_0\}<\frac{1}{\epsilon}V(y_0,r_0).$$

Therefore, by Definition 2.1, it can be seen that the singular system (4) with Markovian jumping parameters is stochastically stable, which implies that the rest point of system (4) is a Nash equilibrium, and this Nash equilibrium is stable. This completes the proof. ■

IV. Illustrative example

In this section, a simple matching pennies zero-sum game example will be given to show the validity of the main result derived in this paper.

Example 4.1: Consider a 3-player matching pennies zero-sum game, three players choose a head (H) or tail (T) of a penny to play the game.

Obversely, there are eight pure strategies, i.e.,

$$S_1=\{T,T,T\},\ S_2=\{H,H,H\},$$
$$S_3=\{T,T,H\},\ S_4=\{T,H,T\},$$
$$S_5=\{H,T,T\},\ S_6=\{H,H,T\},$$
$$S_7=\{H,T,H\},\ S_8=\{T,H,H\}.$$

Similar to [15], we can assume that there are twelve different weight matrices $\widetilde{A}_i=(a^i_{jk})_{3\times3},\widetilde{B}^i_d=(b^i_{jk})_{3\times3},\widetilde{C}^i_g=(c^i_{jk})_{3\times3}$ $(j,k=1,2,3,\ i=1,2,3,4)$ among the eight pure strategies, and assign matrices $\widetilde{A}_1,\widetilde{B}^1_d,\widetilde{C}^1_g$ to strategy S_1, matrices $\widetilde{A}_2,\widetilde{B}^2_d,\widetilde{C}^2_g$ to strategy S_2, matrices $\widetilde{A}_3,\widetilde{B}^3_d,\widetilde{C}^3_g$ to strategy S_3,S_4,S_5, matrices $\widetilde{A}_4,\widetilde{B}^4_d,\widetilde{C}^4_g$ to strategy S_6,S_7,S_8. Then, a delayed zero-sum differential game model with 3 persons can be established as in model (4) with parameters given by

$$\widetilde{A}_1=\left[\begin{array}{ccc}-3&1&1\\1&-4&-2\\1&2&-3\end{array}\right],\widetilde{A}_2=\left[\begin{array}{ccc}-4&1&2\\1&-3&-1\\2&1&-3\end{array}\right],$$

$$\widetilde{A}_3=\left[\begin{array}{ccc}-5&2&2\\-2&-4&1\\1&1&-2\end{array}\right],\widetilde{A}_4=\left[\begin{array}{ccc}-6&4&-1\\-2&-4&1\\2&2&-4\end{array}\right],$$

$$\widetilde{B}^1_d=\left[\begin{array}{ccc}0.1&0.2&0.3\\0.2&0.2&0.1\\0.1&0.1&-0.2\end{array}\right],\widetilde{B}^2_d=\left[\begin{array}{ccc}0.2&0.1&0.2\\0.1&0.1&0.2\\0.1&0.2&-0.3\end{array}\right],$$

$$\widetilde{B}^3_d=\left[\begin{array}{ccc}0.2&0.1&0.1\\0.2&0.2&0.2\\0.3&0.2&-0.5\end{array}\right],\widetilde{B}^4_d=\left[\begin{array}{ccc}0.3&0.1&0.2\\0.2&0.3&0.1\\0.1&0.1&-0.2\end{array}\right],$$

$$\widetilde{C}^1_g=\left[\begin{array}{ccc}\frac{3}{100}&\frac{2}{100}&\frac{1}{100}\\\frac{2}{100}&\frac{2}{100}&\frac{1}{100}\\\frac{1}{100}&\frac{1}{100}&\frac{-2}{100}\end{array}\right],\widetilde{C}^2_g=\left[\begin{array}{ccc}\frac{1}{100}&\frac{4}{100}&\frac{3}{100}\\\frac{2}{100}&\frac{3}{100}&\frac{2}{100}\\0.0&\frac{1}{100}&\frac{-1}{100}\end{array}\right],$$

$$\widetilde{C}^3_g=\left[\begin{array}{ccc}\frac{1}{100}&\frac{1}{100}&\frac{1}{100}\\\frac{2}{100}&\frac{2}{100}&\frac{3}{100}\\\frac{4}{100}&0.0&\frac{-4}{100}\end{array}\right],\widetilde{C}^4_g=\left[\begin{array}{ccc}\frac{2}{100}&\frac{4}{100}&\frac{3}{100}\\\frac{1}{100}&\frac{4}{100}&\frac{2}{100}\\\frac{4}{100}&\frac{2}{100}&\frac{-6}{100}\end{array}\right].$$

If the external perturbation nonlinear function is $g_1(t)=\tanh(-0.3t)$, $g_2(t)=\tanh(-0.8t)$, $g_3(t)=\tanh(-0.1t)$. Obviously, $l_1^-=l_2^-=l_3^-=0,l_1^+=0.3,l_2^+=0.8,l_3^-=0.1$. Considering that the affection caused by players' previous payoffs to current payoffs may be limited, and can become weaker with the developing of game, one can see that time delay should be small, thus we assume $\tau=0.5$, and the transition probability matrix between every four different state weighting matrices is supposed as

$$\Pi=(\pi_{ij})_{4\times4}=\left[\begin{array}{cccc}-0.3&0.2&0.4&-0.3\\-0.1&0.2&-0.4&0.3\\0.2&0.2&0.1&-0.5\\0.4&0.1&0.2&-0.6\end{array}\right].$$

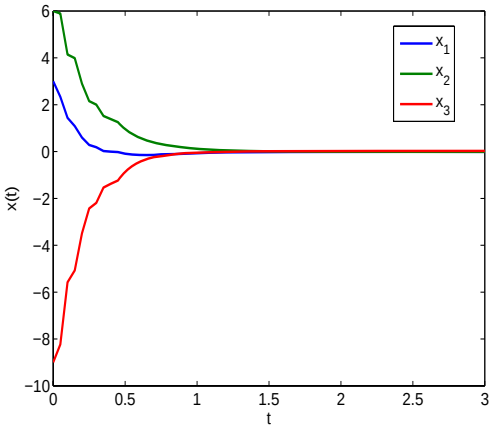


Fig. 2. State variable $x(t)$ of system (1) with initial value $(3, 6, -9)^T$

Notice that $(0, 0, 0)^T$ is a rest point of model (4). If the conditions in Theorem 3.1 are satisfied, the stochastic stability of system (4) can be achieved. In this case, from Remark 2.1, one can see that the states of system (4) have a stable Nash equilibrium at zero. Additional, the conditions given in Theorem 3.1 are all described in the forms of linear matrix inequalities, which can be easily solved by using LMI toolbox in Matlab software through interior point algorithm. The corresponding simulation is given in Fig. 2, from which one can see that the state vector stable to zero, this means that the concerned game has a stable Nash equilibrium.

V. Conclusions

By analyzing the delayed network topology for a zero-sum game, we established a delayed singular game network model with Markovian jumping parameters to describe the relationship among the current payoffs, previous payoffs, and external comprehensive factors. Combined with Finsler's Lemma and Lyapunov stable theory, a sufficient condition guaranteeing the unique existence and stability of zero-sum game's Nash equilibrium is given. Simulation numerical example shows that the result established in this paper is valid.

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