World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:5, No:8, 2011

The diameter of an interval graph is twice of its radius

Tarasankar Pramanik, Sukumar Mondal and Madhumangal Pal

Abstract—In an interval graph $G = (V, E)$ the distance between two vertices u, v is de£ned as the smallest number of edges in a path joining u and v. The eccentricity of a vertex v is the maximum among distances from all other vertices of V. The diameter (δ) and radius (ρ) of the graph G is respectively the maximum and minimum among all the eccentricities of G. The center of the graph G is the set $C(G)$ of vertices with eccentricity ρ. In this context our aim is to establish the relation $\rho = \lceil \frac{\delta}{2} \rceil$ for an interval graph and to determine the center of it.

Keywords—Interval graph, interval tree, radius, center.

I. Introduction

AN undirected graph $G = (V, E)$ is an *interval graph* if the vertex set V can be put into one-to-one correspondence with a set of intervals I on the real line R such that two vertices are adjacent in G if and only if their corresponding intervals have non-empty intersection. The set I is called an *interval representation* of G and G is referred to as the *intersection graph* of I [5]. Let $I = \{i_1, i_2, \ldots, i_n\}$, where $i_c = [a_c, b_c]$ for $1 \leq c \leq n$, be the interval representation of the graph G, a_c is the left endpoint and b_c is the right end point of the interval i_c. Without any loss of generality assumed the following:

(a) an interval contains both its endpoints and that no two intervals share a common endpoint [5],
(b) intervals and vertices of an interval graph are one and the same thing,
(c) the graph G is connected, and the list of sorted endpoints is given and
(d) the intervals in I are indexed by increasing right endpoints, that is, $b_1 < b_2 < \cdots < b_n$.

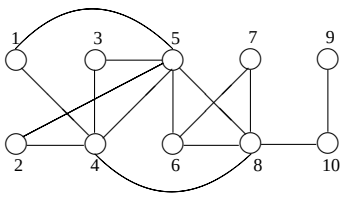


Fig. 1. An interval graph $G = (V, E)$

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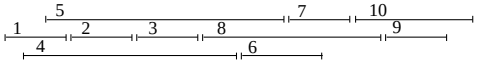


Fig. 2. An interval representation of Figure 1

An interval graph and its interval representation are shown in Figure 1 and Figure 2 respectively.

Interval graphs arise in the process of modeling real life situations, specially involving time dependencies or other restrictions that are linear in nature. This graph and various subclass thereof arise in diverse areas such as archeology, molecular biology, sociology, genetics, traf£c planning, VLSI design, circuit routing, psychology, scheduling, transportation and others. Recently, interval graphs have found applications in protein sequencing [7], macro substitution [2], circuit routine [8], £le organization [1], job scheduling [1], routing of two points nets [6] and many others. An extensive discussion of interval graphs also appears in [5]. Thus interval graphs have been studied intensely from both the theoretical and algorithmic point of view.

The notion of a center in a graph is motivated by a large class of problems collectively referred to as the facility-location problems where one is interested in identifying a subset of the vertices of the graph at which certain facilities are to be located in such a way that for every vertex in the graph, the distance to the nearest facility is minimum.

For a connected graph $G = (V, E)$, the distance $d(u, v)$ between vertices u and v is the smallest number of edges in a path joining u and v.

The *eccentricity* of a vertex $v \in V$, is denoted by $e(v)$ and is de£ned by

$$e(v) = \max\{d(u, v) : u \in V\}.$$

The *diameter* $\delta(G)$ (or simply δ), *radius* $\rho(G)$ (or simply ρ) and the *center* $C(G)$ of a graph G are de£ned as follows:

$$\begin{aligned} \delta(G) &= \max\{e(v) : v \in V\}, \\ \rho(G) &= \min\{e(v) : v \in V\}, \\ \text{and} \quad C(G) &= \{v \in V : e(v) = \rho(G)\}. \end{aligned}$$

The center of a graph may be a single vertex or more than one vertex. This shows in the following £gure. The graph in Figure 3(a) has only one center with center node 2 while the graph in Figure 3(b) has two centers and the center nodes are 3 and 4.

A. Survey of related works

It is both well-known and easy to observe that the center of an arbitrary graph $G = (V, E)$ can be computed by the

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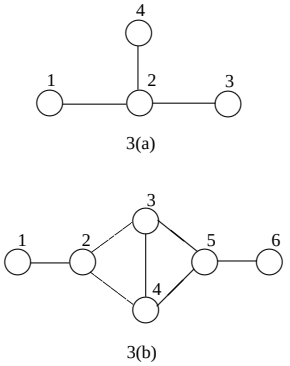


Fig. 3. (a) The center with one vertex; (b) The center with two vertices.

following brute force approach: perform breadth-£rst search of G starting in turn, at every vertex of G. Clearly, this procedure takes $O(|V| \times |E|)$ time.

For some particular classes of graphs, such as for trees [4], outerplaner graphs [3] etc., linear time algorithms can be devised to compute the center. For the interval graph with n vertices and m edges. Olariu [9] has presented an $O(n+m)$ time sequential algorithm where the input is an adjacency list that takes $O(n+m)$ space. In [9], all maximal cliques are used as input. But the generation of all maximal cliques is slightly complicated. In [10], Olariu et al. have presented an $O(\log n)$ time and $O(n)$ processor parallel algorithm to £nd the center of an interval graph if the endpoints list is given. But Pal et. al. [14] have designed optimal algorithm to compute center and diameter of an interval graph, which was an improvement over Olariu's algorithm, since with same input Olariu's algorithm takes $O(n+m)$ time. Also Pal et al. have presented some properties on interval graphs and some properties of diameter and center, but no relation between radius and diameter has been established.

Based on the proposed sequential algorithms two $O(n/P + \log n)$ time parallel algorithms have been designed for the same problems using P processors and $O(n)$ space on the $EREWPRAM$ if the sorted endpoints list is given. It is improvement over the algorithm of [10], because, if the sorted endpoints list is given as an input, then the complexity of the algorithm of [10] remains unchange.

In this paper we have established the relation between the radius and the diameter of the interval graph.

II. Interval tree and its properties

Let $G = (V, E)$, $V = \{1, 2, \ldots, n\}, |V| = n, |E| = m$ be a connected interval graph in which the vertices are given in the sorted order of the right endpoints of the interval representation of the graph. Intervals are labeled according to increasing order of their endpoints. This labeling is referred to as IG ordering. Let (u, v) or (v, u) denote the existence of an adjacency relation between two vertices u, v. It is assumed that (u, u) is always true $i.e.$ $(u, u) \in E$. If $[a_u, b_u]$ and $[a_v, b_v]$ are two end points of the vertices u and v respectively then u, v are adjacent if at least one of the following conditions hold:

(i) $a_v < a_u < b_v$,
(ii) $a_v < b_u < b_v$,
(iii) $a_u < a_v < b_u$,
(iv) $a_u < b_v < b_u$.

The following lemma is true for a given interval graph, $G = (V, E)$.

Lemma 1 ([15]): If the vertices u, v, $w \in V$ be such that $u < v < w$ in the IG-ordering and u is adjacent to w then v is also adjacent to w.

For each vertex $v \in V$ let $H(v)$ represent the highest numbered adjacent vertices of v. If no adjacent vertex of v exists with higher IG number than v then $H(v)$ is assumed to be v.

In other words, $H(v) = \max\{u : (v, u) \in E, u \geq v\}$.

For a given interval graph G, let a tree $T(G) = (V, E')$ be de£ned such that $E' = \{(u, H(u)) : u \in V, u \neq n\}, n$ be the root of $T(G)$. This tree is called the interval tree IT. The various properties of interval tree are available in [11], [12], [14]. The most important property is as follows:

Lemma 2 ([13]): For a connected interval graph there exists a unique interval tree $T(G)$.

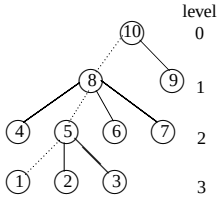


Fig. 4. The interval tree of the interval graph of Figure 2

For each vertex v of interval tree, de£ne $level(v)$ to be the distance of v from the vertex n in the tree.

Let N_l be the set of vertices which are at a distance l from the vertex n. Thus $N_l = \{u : d(u, n) = l\}$ where $d(u, n)$ is the distance between u and n in the interval tree and N_0 is the singleton set $\{n\}$. If $u \in N_l$ then $d(u, n) = l$ and the vertex u is at level l of the interval tree. Thus, the vertices at level l of the interval tree are the vertices of N_l. It follows from Lemma 1, that the vertices of N_l are consecutive integers. Hence the path starting from the vertex 1 and ending at the vertex n in $T(G)$ is called the *main path*. The main path is represented by dotted lines in Figure 4.

De£ne the height h of the tree $T(G)$ by

$$h = \max\{level(v) : v \in V\}.$$

The distance between any two vertices of G can be determined from the following result.

Lemma 3 ([14]): Given $u, v \in V, v \neq n$, let w be the vertex at $level(v) + 1$ on the path from u to n and $w' = H(w)$. If $level(u) > level(v)$, then

$$d(u, v) = \begin{cases} level(u) - level(v), & \text{if } (w, v) \in E \\ level(u) - level(v) + 1, & \text{if } (w, v) \notin E \\ & \quad and\ (w', v) \in E \\ level(u) - level(v) + 2, & \text{otherwise.} \end{cases}$$

The vertex at level l on main path is denoted by v_l^*, l represents the level number and $*$ means it is on the main path.

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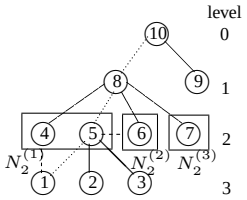


Fig. 5. The partitions of N_2 of the graph shown in Figure 4

III. Diameter

In this section the relation between radius and diameter for the Interval Graph $G = (V, E)$ has been established. In this regarding we recall lemmas 4 to 5.1 was stated in [14].

Lemma 4 ([14]): Let $v_1^* \in N_1$ be the vertex on the main path. If all $v_1 \in N_1$ are adjacent to v_1^* in G then $\delta(G) = h$ otherwise $\delta(G) = h + 1$

If the diameter of the graph G is $h+1$, $h > 1$ then consider one more set of vertices N_{-1} de£ned by

$$N_{-1} = \{u : u \in N_1,\ (u, v_1^*) \notin E,\ \text{where } v_1^* \in N_1\}$$

It is clear that for a vertex $u \in N_{-1}$ if $v_1^* \in N_1$ is the vertex on the main path then the distance between v_1^* and u is 2, since there exists only one path from v_1^* to u that passes through the vertex $n \in N_0$.

Let v_l be any vertex at level l and v_{l+1}^* be the vertex at level $l+1$ on the main path. We recall two parameters d_1 and d_{-1} from [14]. They are de£ned as

$$d_1 = \begin{cases} h - l, & \text{if } (v_l, v_{l+1}^*) \in E \\ h - l + 1, & \text{if } (v_l, v_{l+1}^*) \notin E,\ (v_l, v_l^*) \in E \\ h - l + 2, & \text{otherwise} \end{cases}$$

and

$$d_{-1} = \begin{cases} l+1, \text{ if } (v_l, v_2) \notin E \text{ and } (v_1^*, v_1) \notin E \text{ for all} \\ \qquad v_1 \in N_1, v_2 \in N_2 \text{ on the path from } v_l \text{ to } n \\ l, \quad \text{otherwise} , \end{cases}$$
$$= \begin{cases} l+1, \text{ if } N_{-1} \neq \Phi \\ l, \quad \text{if } N_{-1} = \Phi. \end{cases}$$

Lemma 5 ([14]): Let v_l be a vertex at level l and v_l^* be the vertex at the same level on the main path. The maximum distance $d_{\max}(v_l)$ is given by

$$d_{\max}(v_l) = \max\{d(u, v_l) : u \in V\} = \max(d_1, d_{-1}).$$

The center of the graph G is denoted by $C(G)$. An explicit form to compare center of G is given below.

Corollary 5.1 ([14]): If $d_{\max}(v_l) = \rho$ then $v_l \in C(G)$.

Now let us partition the vertices at level l of the interval tree $T(G)$ into three disjoint subsets $N_l^{(1)}, N_l^{(2)}$ and $N_l^{(3)}$ as the following way:

$$N_l^{(1)} = \{v_l : (v_l, v_{l+1}^*) \in E\}$$
$$N_l^{(2)} = \{v_l : (v_l, v_{l+1}^*) \notin E, (v_l, v_l^*) \in E\}$$
$$N_l^{(3)} = \{v_l : (v_l, v_{l+1}^*) \notin E, (v_l, v_l^*) \notin E\}.$$

Then,

$$d_1 = \begin{cases} h - l, & \text{if } v_l \in N_l^{(1)} \\ h - l + 1, & \text{if } v_l \in N_l^{(2)} \\ h - l + 2, & \text{if } v_l \in N_l^{(3)} \end{cases}$$

Therefore the eccentricity of the vertex, radius, diameter and center of the graph can be given by the following manner:

$$\begin{aligned} e(v_l) &= \max\{d(u, v_l) : \forall\, u \in V\} \\ &= d_{\max}(v_l) = \max(d_1, d_{-1}), \\ \rho(G) &= \min\{e(v) : v \in V\}, \\ \delta(G) &= \max\{e(v) : v \in V\} \\ &= \begin{cases} h, & \text{if } N_{-1} = \Phi \\ h+1, & \text{if } N_{-1} \neq \Phi, \end{cases} \\ C(G) &= \{v : e(v) = \rho, v \in V\} \\ &= \{v_l : d_{\max}(v_l) = \rho\}. \end{aligned}$$

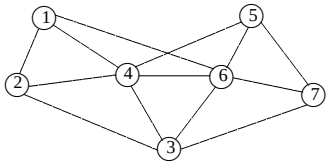


Fig. 6. An arbitrary graph $G_A = (V_A, E_A)$ in which $\rho \neq \lceil \frac{\delta}{2} \rceil$

TABLE I
Distances between vertices for the graph of Figure 6

v_A	1	2	3	4	5	6	7
1	0	1	2	1	2	1	2
2	1	0	1	1	2	2	2
3	2	1	0	1	2	1	1
4	1	1	1	0	1	1	2
5	2	2	2	1	0	1	1
6	1	2	1	1	1	0	1
7	2	2	1	2	1	1	0

TABLE II
Eccentricities of the vertices for the graph of Figure 6

v_A	1	2	3	4	5	6	7
$e(v_A)$	2	2	2	2	2	2	2

Next we investigate the relation between radius and diameter of an interval graph. In general, the relation $\rho = \lceil \frac{\delta}{2} \rceil$ is not valid for an arbitrary graph. We argue this statement by considering a counter example. For this purpose we consider the graph shown in Figure 6. Then we compute the shortest distances and eccentricities and put them in the following two tables Table I and Table II respectively for all the vertices of graph of Figure 6.

From these two tables it is easily seen that $\delta(G_A) = \max\{e(v_A) : v_A \in V_A\} = 2$ and $\rho(G_A) = \min\{e(v_A) : v_A \in V_A\} = 2$, and then $\rho(G_A) \neq \left\lceil \frac{\delta(G_A)}{2} \right\rceil$.

Lemma 6: For a given interval graph G, $\rho = \lceil \frac{\delta}{2} \rceil$ and the center $C(G)$ of the graph G is given by

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(i) when $\delta > 1$ then

$$C(G) = \begin{cases} \left\{v : v \in N_l^{(1)},\ l = \left\lceil \frac{h-1}{2} \right\rceil \text{ or } \left\lfloor \frac{h-1}{2} \right\rfloor \right\}, \\ \qquad \text{if } h \text{ is odd and } N_{-1} \neq \Phi \\ \left\{v : v \in N_l^{(1)} \cup N_l^{(2)},\ l = \left\lceil \frac{h}{2} \right\rceil \text{ or } \left\lfloor \frac{h-1}{2} \right\rfloor \right\}, \\ \qquad \text{if } h \text{ is even and } N_{-1} \neq \Phi \\ \left\{v : v \in N_l^{(1)},\ l = \left\lceil \frac{h}{2} \right\rceil \text{ or } \left\lfloor \frac{h}{2} \right\rfloor \right\}, \\ \qquad \text{if } h \text{ is even and } N_{-1} = \Phi \\ \left\{v : v \in N_l^{(1)} \cup N_l^{(2)},\ l = \left\lceil \frac{h+1}{2} \right\rceil \text{ or } \left\lfloor \frac{h}{2} \right\rfloor \right\}, \\ \qquad \text{if } h \text{ is odd and } N_{-1} = \Phi, \end{cases}$$

(ii) when $\delta = 1$ then $C(G) = V$.

Proof. *(i)* Suppose $N_{-1} \neq \Phi$. Then $d_{-1} = l + 1$ (by Lemma 4) and we have the following three cases:

Case 1: If $v \in N_l^{(1)}$, $d_1 = h - l$. Then,

$$\begin{aligned} e(v) &= \max\{h - l, l + 1\} \\ &= \begin{cases} h - l, \text{ for } l \leq \left\lfloor \frac{h-1}{2} \right\rfloor \\ l + 1, \text{ for } l \geq \left\lceil \frac{h-1}{2} \right\rceil \end{cases}. \end{aligned}$$

Therefore, $e_1 = \min\{e(v) : v \in N_l^{(1)}\} = \left\lceil \frac{h+1}{2} \right\rceil$, for $l = \left\lfloor \frac{h-1}{2} \right\rfloor$ or $\left\lceil \frac{h-1}{2} \right\rceil$.

Case 2: If $v \in N_l^{(2)}$, $d_1 = h - l + 1$. Then,

$$\begin{aligned} e(v) &= \max\{h - l + 1, l + 1\} \\ &= \begin{cases} h - l + 1, & \text{for } l \leq \left\lfloor \frac{h}{2} \right\rfloor \\ l + 1, & \text{for } l \geq \left\lceil \frac{h}{2} \right\rceil \end{cases}. \end{aligned}$$

Therefore, $e_2 = \min\{e(v) : v \in N_l^{(2)}\} \left\lceil \frac{h+2}{2} \right\rceil$, for $l = \left\lfloor \frac{h}{2} \right\rfloor$ or $\left\lceil \frac{h}{2} \right\rceil$.

Case 3: If $v \in N_l^{(3)}$, $d_1 = h - l + 2$. Then,

$$\begin{aligned} e(v) &= \max\{h - l + 2, l + 1\} \\ &= \begin{cases} h - l + 2, & \text{for } l \leq \left\lfloor \frac{h+1}{2} \right\rfloor \\ l + 1, & \text{for } l \geq \left\lceil \frac{h+1}{2} \right\rceil \end{cases}. \end{aligned}$$

Therefore, $e_3 = \min\{e(v) : v \in N_l^{(3)}\} \left\lceil \frac{h+3}{2} \right\rceil$, for $l = \left\lfloor \frac{h+1}{2} \right\rfloor$ or $\left\lceil \frac{h+1}{2} \right\rceil$.

Hence combining above three cases we have,

$$\begin{aligned} \min\{e(v) : v \in N_l\} &= \min\{e_1, e_2, e_3\} \\ &= \min\left\{\left\lceil \frac{h+1}{2} \right\rceil, \left\lceil \frac{h+2}{2} \right\rceil, \left\lceil \frac{h+3}{2} \right\rceil\right\} \\ &= \min\left\{\left\lceil \frac{h+1}{2} \right\rceil, \left\lceil \frac{h+2}{2} \right\rceil\right\} \\ &= \begin{cases} \left\lceil \frac{h+1}{2} \right\rceil, \text{ for } v \in N_l^{(1)},\ l = \left\lfloor \frac{h-1}{2} \right\rfloor \\ \qquad \text{or } \left\lceil \frac{h-1}{2} \right\rceil \text{ and } h \text{ is odd} \\ \left\lceil \frac{h+1}{2} \right\rceil, \text{ for } v \in N_l^{(2)},\ l = \left\lceil \frac{h}{2} \right\rceil \\ \qquad \text{or } \left\lfloor \frac{h-1}{2} \right\rfloor \text{ and } h \text{ is even.} \end{cases} \end{aligned}$$

Thus, in this case $\rho = \left\lceil \frac{\delta}{2} \right\rceil$ where,

$$C(G) = \begin{cases} \left\{v : v \in N_l^{(1)}\right\}, & \text{for } l = \left\lfloor \frac{h-1}{2} \right\rfloor \text{ or } \left\lceil \frac{h-1}{2} \right\rceil \\ & \text{and } h \text{ is odd} \\ \left\{v : v \in N_l^{(1)} \cup N_l^{(2)}\right\}, & \text{for } l = \left\lceil \frac{h}{2} \right\rceil \text{ or } \left\lfloor \frac{h-1}{2} \right\rfloor \\ & \text{and } h \text{ is even.} \end{cases}$$

Next when $N_{-1} = \Phi$. Then $d_{-1}(v) = l$ (by Lemma 4). Again there are three cases.

Case 1: If $v \in N_l^{(1)}$, $d_1 = h - l$. Then,

$$\begin{aligned} e(v) &= \max\{h - l, l\} \\ &= \begin{cases} h - l, & \text{for } l \leq \left\lfloor \frac{h}{2} \right\rfloor \\ l, & \text{for } l \geq \left\lceil \frac{h}{2} \right\rceil \end{cases}. \end{aligned}$$

Therefore, $e_1' = \min\{e(v) : v \in N_l^{(1)}\} \left\lceil \frac{h}{2} \right\rceil$, for $l = \left\lfloor \frac{h}{2} \right\rfloor$ or $\left\lceil \frac{h}{2} \right\rceil$.

Case 2: If $v \in N_l^{(2)}$, $d_l(v) = h - l + 1$. Then,

$$\begin{aligned} e(v) &= \max\{h - l + 1, l\} \\ &= \begin{cases} h - l + 1, & \text{for } l \leq \left\lfloor \frac{h+1}{2} \right\rfloor \\ l, & \text{for } l \geq \left\lceil \frac{h+1}{2} \right\rceil \end{cases}. \end{aligned}$$

Therefore, $e_2' = \min\{e(v) : v \in N_l^{(2)}\} \left\lceil \frac{h+1}{2} \right\rceil$, for $l = \left\lfloor \frac{h+1}{2} \right\rfloor$ or $\left\lceil \frac{h+1}{2} \right\rceil$.

Case 3: If $v \in N_l^{(3)}$, $d_l(v) = h - l + 2$. Then,

$$\begin{aligned} e(v) &= \max\{h - l + 2, l\} \\ &= \begin{cases} h - l + 2, & \text{for } l \leq \left\lfloor \frac{h+2}{2} \right\rfloor \\ l, & \text{for } l \geq \left\lceil \frac{h+2}{2} \right\rceil \end{cases}. \end{aligned}$$

Therefore, $e_3' = \min\{e(v) : v \in N_l^{(3)}\} \left\lceil \frac{h+2}{2} \right\rceil$, for $l = \left\lfloor \frac{h+2}{2} \right\rfloor$ or $\left\lceil \frac{h+2}{2} \right\rceil$.

Combining these three cases we have,

$$\begin{aligned} \min\{e(v) : v \in N_l\} &= \min\{e_1', e_2', e_3'\} \\ &= \min\left\{\left\lceil \frac{h}{2} \right\rceil, \left\lceil \frac{h+1}{2} \right\rceil, \left\lceil \frac{h+2}{2} \right\rceil\right\} \\ &= \begin{cases} \left\lceil \frac{h}{2} \right\rceil, & \text{for } v \in N_l^{(1)},\ l = \left\lfloor \frac{h}{2} \right\rfloor \\ & \text{or } \left\lceil \frac{h}{2} \right\rceil \text{ and } h \text{ is even} \\ \left\lceil \frac{h}{2} \right\rceil, & \text{for } v \in N_l^{(2)},\ l = \left\lceil \frac{h+1}{2} \right\rceil \\ & \text{or } \left\lfloor \frac{h}{2} \right\rfloor \text{ and } h \text{ is odd.} \end{cases} \end{aligned}$$

Thus, in this case $\rho = \left\lceil \frac{\delta}{2} \right\rceil$ where,

$$C(G) = \begin{cases} \left\{v : v \in N_l^{(1)}\right\}, & \text{for } l = \left\lfloor \frac{h}{2} \right\rfloor \text{ or } \left\lceil \frac{h}{2} \right\rceil \\ & \text{and } h \text{ is even} \\ \left\{v : v \in N_l^{(1)} \cup N_l^{(2)}\right\}, & \text{for } l = \left\lceil \frac{h+1}{2} \right\rceil \text{ or } \left\lfloor \frac{h}{2} \right\rfloor \\ & \text{and } h \text{ is odd.} \end{cases}$$

Hence whatever the case may be, $\rho = \left\lceil \frac{\delta}{2} \right\rceil$ and center of the graph be

$$C(G) = \begin{cases} \left\{v : v \in N_l^{(1)},\ l = \left\lceil \frac{h-1}{2} \right\rceil \text{ or } \left\lfloor \frac{h-1}{2} \right\rfloor \right\}, \\ \qquad \text{if } h \text{ is odd and } N_{-1} \neq \Phi \\ \left\{v : v \in N_l^{(1)} \cup N_l^{(2)},\ l = \left\lceil \frac{h}{2} \right\rceil \text{ or } \left\lfloor \frac{h-1}{2} \right\rfloor \right\}, \\ \qquad \text{if } h \text{ is even and } N_{-1} \neq \Phi \\ \left\{v : v \in N_l^{(1)},\ l = \left\lceil \frac{h}{2} \right\rceil \text{ or } \left\lfloor \frac{h}{2} \right\rfloor \right\}, \\ \qquad \text{if } h \text{ is even and } N_{-1} = \Phi \\ \left\{v : v \in N_l^{(1)} \cup N_l^{(2)},\ l = \left\lceil \frac{h+1}{2} \right\rceil \text{ or } \left\lfloor \frac{h}{2} \right\rfloor \right\}, \\ \qquad \text{if } h \text{ is odd and } N_{-1} = \Phi, \end{cases}$$

(ii) It is clear that if $\delta = 1$ then there are atmost two vertices in G. So, $C(G) = V$. Then $\rho = 1 = \left\lceil \frac{1}{2} \right\rceil = \left\lceil \frac{\delta}{2} \right\rceil$. Hence the lemma.

Also the center of the graph is given by the following result without use of ρ.

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Lemma 7: The center of a connected interval graph $G = (V, E)$ is given by

$$C(G) = \begin{cases} \left\{v : v \in N_l^{(1)} \cup N_{r+1}^{(1)} \cup N_{r+1}^{(2)}\right\}, & \text{if } \delta \text{ is odd} \\ \left\{v : v \in N_l^{(1)}\right\}, & \text{if } \delta \text{ is even} \end{cases}$$

where,

$$r = \begin{cases} \left\lfloor \frac{\delta}{2} \right\rfloor, & \text{if } N_{-1} = \Phi \\ \left\lfloor \frac{\delta}{2} \right\rfloor - 1, & \text{if } N_{-1} \neq \Phi. \end{cases}$$

Proof. The center of the graph G obtained by lemma 6 is

$$C(G) = \begin{cases} \left\{v : v \in N_l^{(1)},\ l = \left\lceil \frac{h-1}{2} \right\rceil \text{ or } \left\lfloor \frac{h-1}{2} \right\rfloor\right\}, \\ \qquad \text{if } h \text{ is odd and } N_{-1} \neq \Phi \\ \left\{v : v \in N_l^{(1)} \cup N_l^{(2)},\ l = \left\lceil \frac{h}{2} \right\rceil \text{ or } \left\lfloor \frac{h-1}{2} \right\rfloor\right\}, \\ \qquad \text{if } h \text{ is even and } N_{-1} \neq \Phi \\ \left\{v : v \in N_l^{(1)},\ l = \left\lceil \frac{h}{2} \right\rceil \text{ or } \left\lfloor \frac{h}{2} \right\rfloor\right\}, \\ \qquad \text{if } h \text{ is even and } N_{-1} = \Phi \\ \left\{v : v \in N_l^{(1)} \cup N_l^{(2)},\ l = \left\lceil \frac{h+1}{2} \right\rceil \text{ or } \left\lfloor \frac{h}{2} \right\rfloor\right\}, \\ \qquad \text{if } h \text{ is odd and } N_{-1} = \Phi, \end{cases}$$

Let

$$r = \begin{cases} \left\lfloor \frac{\delta}{2} \right\rfloor, & \text{if } N_{-1} = \Phi \\ \left\lfloor \frac{\delta}{2} \right\rfloor - 1, & \text{if } N_{-1} \neq \Phi. \end{cases}$$

Then we have two cases:
Case 1: The case when $N_{-1} \neq \Phi$, then $\delta = h + 1$. Therefore,

$$\begin{aligned} l &= \left\lceil \tfrac{h-1}{2} \right\rceil \text{ or } \left\lfloor \tfrac{h-1}{2} \right\rfloor \\ &= \left\lceil \tfrac{\delta-2}{2} \right\rceil \text{ or } \left\lfloor \tfrac{\delta-2}{2} \right\rfloor \\ &= \left\lceil \tfrac{\delta}{2} \right\rceil - 1 \text{ or } \left\lfloor \tfrac{\delta}{2} \right\rfloor - 1 \\ &= \begin{cases} \left\lfloor \frac{\delta}{2} \right\rfloor - 1 \text{ or } \left\lfloor \frac{\delta}{2} \right\rfloor - 1, & \text{if } \delta \text{ is even} \\ \left\lceil \frac{\delta}{2} \right\rceil \text{ or } \left\lfloor \frac{\delta}{2} \right\rfloor - 1, & \text{if } \delta \text{ is odd} \end{cases} \\ &= \begin{cases} \left\lfloor \frac{\delta}{2} \right\rfloor - 1, & \text{if } \delta \text{ is even} \\ \left\lceil \frac{\delta}{2} \right\rceil \text{ or } \left\lfloor \frac{\delta}{2} \right\rfloor - 1, & \text{if } \delta \text{ is odd} \end{cases} \\ &= \begin{cases} r, & \text{if } \delta \text{ is even} \\ r \text{ or } r+1, & \text{if } \delta \text{ is odd.} \end{cases} \end{aligned}$$

Therefore,

$$v \in \begin{cases} N_r^{(1)} \cup N_{r+1}^{(1)}, & \text{if } \delta \text{ is odd} \\ N_r^{(1)}, & \text{if } \delta \text{ is even.} \end{cases}$$

Thus it is clear that when h is even, δ is odd. Then,

$$\begin{aligned} l &= \left\lceil \tfrac{h}{2} \right\rceil \\ &= \left\lceil \tfrac{\delta-1}{2} \right\rceil \\ &= \left\lceil \tfrac{\delta}{2} \right\rceil \\ &= r + 1. \end{aligned}$$

Therefore, $v \in N_{r+1}^{(2)}$, if δ is odd. Hence,

$$C(G) = \begin{cases} \left\{v : v \in N_l^{(1)} \cup N_{r+1}^{(1)} \cup N_{r+1}^{(2)}\right\}, & \text{if } \delta \text{ is odd} \\ \left\{v : v \in N_l^{(1)}\right\}, & \text{if } \delta \text{ is even.} \end{cases}$$

Case 2: The case when $N_{-1} = \Phi$, then $\delta = h$. Therefore,

$$\begin{aligned} l &= \left\lceil \tfrac{h}{2} \right\rceil \text{ or } \left\lfloor \tfrac{h}{2} \right\rfloor \\ &= \left\lceil \tfrac{\delta}{2} \right\rceil \text{ or } \left\lfloor \tfrac{\delta}{2} \right\rfloor \\ &= \left\lceil \tfrac{\delta}{2} \right\rceil - 1 \text{ or } \left\lfloor \tfrac{\delta}{2} \right\rfloor - 1 \\ &= \begin{cases} \left\lfloor \frac{\delta}{2} \right\rfloor \text{ or } \left\lfloor \frac{\delta}{2} \right\rfloor + 1, & \text{if } \delta \text{ is even} \\ \left\lceil \frac{\delta}{2} \right\rceil \text{ or } \left\lceil \frac{\delta}{2} \right\rceil, & \text{if } \delta \text{ is odd} \end{cases} \\ &= \begin{cases} r, & \text{if } \delta \text{ is even} \\ r \text{ or } r+1, & \text{if } \delta \text{ is odd.} \end{cases} \end{aligned}$$

Therefore,

$$v \in \begin{cases} N_r^{(1)} \cup N_{r+1}^{(1)}, & \text{if } \delta \text{ is odd} \\ N_r^{(1)}, & \text{if } \delta \text{ is even.} \end{cases}$$

Now it is clear that when h is odd, δ is odd. Then,

$$\begin{aligned} l &= \left\lceil \tfrac{h+1}{2} \right\rceil \\ &= \left\lceil \tfrac{\delta+1}{2} \right\rceil \\ &= \left\lceil \tfrac{\delta}{2} \right\rceil + 1 \\ &= r + 1. \end{aligned}$$

Therefore, $v \in N_{r+1}^{(2)}$, if δ is odd. Hence,

$$C(G) = \begin{cases} \left\{v : v \in N_l^{(1)} \cup N_{r+1}^{(1)} \cup N_{r+1}^{(2)}\right\}, & \text{if } \delta \text{ is odd} \\ \left\{v : v \in N_l^{(1)}\right\}, & \text{if } \delta \text{ is even.} \end{cases}$$

IV. Conclusion

In this paper some properties of an interval graph are introduced. We have worked to prove a relation $\rho = \lceil \frac{\delta}{2} \rceil$. Also, the center of an interval graph has been calculated without use of ρ. We think it will enrich all most all researchers.

References

[1] M. C. Carlisle and E. L. Loyd, On the *k*-coloring of intervals, *LNCS, ICCI*, 497 (1991) 90-101.
[2] J. Fabri, *Automatic Storage Optimization*, (UMI Press Ann Arbor, MI), 1982.
[3] A. Farley and A. Proskurowski, Computation of the center and diameter of Outerplanar graphs, *Discrete Applied Mathematics*, 2 (1980) 185-191.
[4] A. J. Goldman, MInimax location of a facility in a network, *Transportation Science*, 6 (1972) 407-418.
[5] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York, 1980.
[6] A. Hashimoto and J. Stevens, Wire routing by optimizing cannel assignment within large apertures, *Proc., 8th IEEE Design Automation Workshop*, (1971) 155-169.
[7] J. R. Jungck, O. Dick, and A. G. Dick, Computer assisted sequencing, interval graphs and molecular evolution, *Biosystem*, 15 (1982) 259-273.
[8] T. Ohtsuki, H. Mori E. S. Khu, T. Kashiwabara and T. Fujisawa, One dimensional logic gate assignment and interval graph, *IEEE Trans. Circuits and Systems*, 26 (1979) 675-684.
[9] S. Olariu, A simple linear time algorithm for compting the center of an interval grap, *International Journal of Computer Mathematics*, 34 (1990) 121-128.
[10] S. Olariu, J. L. Schwing and J. Zhang, Optimal parallel algoritms for problems modeled by a family of intervals, *IEEE Trans. Paralles and Distributed Systems*, 3 (1992) 364-374.
[11] A. Pal and M. Pal, Interval tree and its applications, *Advanced Modeling and Optimization (An Electronic International Journal)*, 11(3) (2009) 211-224.
[12] M. Pal, *Some sequential and parallel algorithms on interval graphs*, Ph.D Thesis, Indian Institute of Technology, Kharagpur, India, 1995.
[13] M. Pal and G. P. Bhattecharjee, An optimal parallel algorithm to solve the all pair shortest path problem on an interval graph, *In proceeding of 3rd National Seminar on theoritical computer science, India*, June (1993) 309-318.

[14] M. Pal, G. P. Bhattacharjee, Optimal sequential and parallel algorithms for computing the diameter and the center of an interval graph, *International Journal of Computer Mathematics*, 59 (1995) 1-13.
[15] G. Ramalingam and C. Pandu Rangan, A uni£ed approach to domination problems in interval graphs, *Information Processing Letters*, 27 (1988) 271-274.

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