

Continuity of Defuzzification and Its Application to Fuzzy Control

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Abstract—The mathematical framework for studying of a fuzzy approximate reasoning is presented in this paper. Two important defuzzification methods (Area defuzzification and Height defuzzification) besides the center of gravity method which is the best well known defuzzification method are described. The continuity of the defuzzification methods and its application to a fuzzy feedback control are discussed.

Keywords—Fuzzy approximate reasoning, defuzzification, area method, height method.

I. INTRODUCTION

ZADEH introduced the fuzzy set expanded the usual set (crisp set) and constructed the fuzzy theory that fuzziness by human language can be treated quantitatively [1]. Mamdani has applied it to the field of control theory [2]. Fuzzy logic control is one of the ways to represent numerically the control given by human language and sensitivity, and it has been applied in various practical control plants. Since 1990's, like the theory of classical control and modern control, many systematized mathematical considerations have been discussed [3] [4]. Tanaka developed the approach to the stability analysis for fuzzy control by means of Lyapunov's stability method [5]. Hojo discussed the macroscopic stability of fuzzy control systems [6]. Furuhashi analyzed the stability of fuzzy control system represented by petri net [7]. In practical use, fuzzy membership functions (fuzzy sets), which represent input and output states in optimal control system, are decided on the basis of the experience of experts in each peculiar plant before. Therefore some acquisition methods of fuzzy inference rules by a neural network and a genetic algorithm have been proposed [8] [9].

The authors have been studying to establish the automatic and computational determination of fuzzy membership functions, which give optimal controls in fuzzy control system [10]. The authors also have been studying to find algorithms that compute optimal solutions. The authors consider fuzzy optimal control problems as problems of finding the minimum (maximum) value of the cost (benefit) function with feedback law constructed by Mamdani method, product-sum-gravity method, and Nakamori method [11]–[13]. These approximate reasoning methods adopt the center of gravity method, and calculate defuzzified value of inference result represented by fuzzy set. This defuzzification method is most widely used.

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The resulting behavior of fuzzy approximate reasoning using any of these defuzzification methods will be discussed in the following section. The author's study covers two defuzzification methods: Area method and Height method [14] [19]. Since these methods do not synthesize the fuzzy set (membership function), they are better than the center of gravity method in respect of high-speed computing.

In this study, two kinds of continuity of defuzzification are discussed. One is Lipschitz continuity on the space of premise valuable. The other is continuity as functional on the set of membership functions. By the continuity as functional and the compactness of the set of membership functions, the existence of an optimal feedback control law in a nonlinear fuzzy feedback control system, in which the feedback laws are determined by IF-THEN type fuzzy rules, are shown. Then it is crucial to investigate the convergence of feedback laws constructed by fuzzy approximate reasoning method and the convergence of solutions of the nonlinear state equation in the fuzzy control system.

II. FUZZY FEEDBACK CONTROL

\mathbb{R}^n denotes the n -dimensional Euclidean space with the usual norm $\|\cdot\|$. Let $f(y, v) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ be a (nonlinear) vector valued function which is Lipschitz continuous. In addition, assume that there exists a constant $M_f > 0$ such that

$$\|f(y, v)\| \leq M_f (\|y\| + |v| + 1) \quad (1)$$

for all $(y, v) \in \mathbb{R}^n \times \mathbb{R}$.

Consider a system given by the following state equation:

$$\dot{x}(t) = f(x(t), u(t)), \quad (2)$$

where $x(t)$ is the state and the control input $u(t)$ of the system is given by the state feedback

$$u(t) = \rho(x(t)).$$

Assume that the controllability is guaranteed in this system. For a sufficiently large $r > 0$,

$$B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$$

denotes a bounded set containing all possible initial states x_0 of the system. Let T be a sufficiently large final time. Then, we have

Proposition [11]. Let $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Lipschitz continuous function and $x_0 \in B_r$. Then, the state equation

$$\dot{x}(t) = f(x(t), \rho(x(t))) \quad (3)$$

has a unique solution $x(t, x_0, \rho)$ on $[0, T]$ with the initial condition $x(0) = x_0$ such that the mapping $(t, x_0) \in [0, T] \times B_r \mapsto x(t, x_0, \rho)$ is continuous.

For any $r_2 > 0$, put

$$\Phi = \{\rho: \mathbb{R}^n \rightarrow \mathbb{R} : \text{Lipschitz continuous,} \\ \sup_{u \in \mathbb{R}^n} |\rho(u)| \leq r_2\}. \quad (4)$$

Then, the following (a) and (b) hold.

(a) For any $t \in [0, T]$, $x_0 \in B_r$ and $\rho \in \Phi$,

$$\|x(t, x_0, \rho)\| \leq r_1, \quad (5)$$

where

$$r_1 = e^{M_f T} r + (e^{M_f T} - 1)(r_2 + 1). \quad (6)$$

(b) Let $\rho_1, \rho_2 \in \Phi$. Then, for any $t \in [0, T]$ and $x_0 \in B_r$,

$$\|x(t, x_0, \rho_1) - x(t, x_0, \rho_2)\| \\ \leq \frac{e^{L_f(1+L_{\rho_1})t} - 1}{1 + L_{\rho_1}} \sup_{u \in [-r_1, r_1]^n} |\rho_1(u) - \rho_2(u)|, \quad (7)$$

where L_f and L_{ρ_1} are the Lipschitz constants of f and ρ_1 .

III. FUZZY RULES AND FUZZY SETS

Assume the feedback law ρ consists of the following m IF-THEN type fuzzy control rules.

RULE 1: IF x_1 is A_{11} and ... and x_n is A_{1n}

THEN y is B_1
⋮

RULE i : IF x_1 is A_{i1} and ... and x_n is A_{in}

THEN y is B_i
⋮

RULE m : IF x_1 is A_{m1} and ... and x_n is A_{mn}

THEN y is B_m (8)

Here, m is the number of fuzzy production rules, and n is the number of premise variables $x_j \in [-r_1, r_1]$. $y \in [-r_2, r_2]$ is consequence variable. The constants r_1 and r_2 are decided by (4) and (6) in the previous section. Let $\mu_{A_{ij}}$ and μ_{B_i} be membership functions of the fuzzy set A_{ij} and B_i , respectively. Let $C[-r_1, r_1]$ and $C[-r_2, r_2]$ be the Banach space of all continuous real functions. The following two sets of fuzzy membership functions:

$$F_{\Delta_{ij}} = \{\mu \in C[-r_1, r_1] : \forall x \in [-r_1, r_1] \ 0 \leq \mu(x) \leq 1,$$

$$\forall x, x' \in [-r_1, r_1] \ |\mu(x) - \mu(x')| \leq \Delta_{ij}|x - x'|\} \quad (9)$$

and

$$G = \{\mu \in C[-r_2, r_2] : \forall x \in [-r_2, r_2] \ 0 \leq \mu(x) \leq 1\} \quad (10)$$

are considered. Assume that $\mu_{A_{ij}}$ and μ_{B_i} are element of $F_{\Delta_{ij}}$ and G , respectively. Put

$$\mathcal{F} = \prod_{i=1}^m \left\{ \prod_{j=1}^n F_{\Delta_{ij}} \right\} \times G^m \quad (11)$$

where G^m denotes the m -times Cartesian product of G . For simplicity, we write “IF” and “THEN” parts in the rules by the following notation:

$$\mathcal{A}_i = (\mu_{A_{i1}}, \mu_{A_{i2}}, \dots, \mu_{A_{in}}) \ (i = 1, 2, \dots, m),$$

$$\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m) \text{ and } \mathcal{B} = (\mu_{B_1}, \mu_{B_2}, \dots, \mu_{B_m}).$$

The membership function usually used in the fuzzy control and defined on the real number space is triangular, trapezoidal (π -type), bell-shaped, Z-type, and S-type, etc. If closed interval $[-r_1, r_1]$ and $[-r_2, r_2]$ are taken sufficiently large in a practical use, almost all membership functions are included in the set defined by (9) and (10).

Then, the IF-THEN type fuzzy control rules above is called a fuzzy controller, and is denoted by $(\mathcal{A}, \mathcal{B})$. In the rules, the tuple of premise variable $x = x_1, x_2, \dots, x_n$ is called an input information given to the fuzzy controller $(\mathcal{A}, \mathcal{B})$, and y is called a control variable.

IV. APPROXIMATE REASONING

Let $(\mathcal{A}, \mathcal{B})$ be a fuzzy controller given by the IF-THEN type fuzzy control rules above. We say that the system (3) is a *fuzzy feedback system* if the control function $u(t)$ is given by the state feedback $u(t) = \rho_{\mathcal{A}\mathcal{B}}(x(t))$, where $\rho_{\mathcal{A}\mathcal{B}}(x(t))$ is the amount of operation from the fuzzy controller $(\mathcal{A}, \mathcal{B})$ for the input information $x(t)$. Mamdani method is widely used in fuzzy controls because of its simplicity and comprehensibility. Since min-max gravity method uses minimum and maximum operations, because of their nonlinearity, the value of the agreement degree and the gravity might not change smoothly. In addition, it is pointed out that this method is not necessarily appropriate to express the human intuition. Then, Mizumoto proposed the product-sum-gravity method by replacing minimum with product and maximum with summation [15]. In the following, the approximate reasoning methods with area defuzzification method and height defuzzification method are introduced. If given the input information $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ to the fuzzy controller $(\mathcal{A}, \mathcal{B})$, the procedure for inference is summarized as follows:

Procedure 1. The strength of each rule is calculated by

$$\alpha_{m\mathcal{A}_i}(x^*) = \bigwedge_{j=1}^n \mu_{A_{ij}}(x_j^*),$$

$$\alpha_{p\mathcal{A}_i}(x^*) = \prod_{j=1}^n \mu_{A_{ij}}(x_j^*) \quad (i = 1, 2, \dots, m).$$

Procedure 2. The control output of the each rule is calculated by

$$\beta_{m\mathcal{A}_i\mathcal{B}_i}(x^*, y) = \alpha_{\mathcal{A}_i}(x^*) \wedge \mu_{B_i}(y),$$

$$\beta_{p\mathcal{A}_i\mathcal{B}_i}(x^*, y) = \alpha_{\mathcal{A}_i}(x^*) \cdot \mu_{B_i}(y) \quad (i = 1, 2, \dots, m)$$

where $\alpha_{\mathcal{A}_i}$ is $\alpha_{m\mathcal{A}_i}$ or $\alpha_{p\mathcal{A}_i}$, and $\{\cdot\}$ indicates multiplication.

Procedure 3. The defuzzified value y_i^* of the fuzzy set in consequent part of i -th rule is given by

$$y_i^* = \frac{\int_{-r_2}^{r_2} y \mu_{B_i}(y) dy}{\int_{-r_2}^{r_2} \mu_{B_i}(y) dy} \quad (i = 1, 2, \dots, m).$$

Procedure 4. Defuzzification stage.

Area method:

$$\rho_{aAB}(x^*) = \frac{\sum_{i=1}^m S_{A_i B_i}(x^*) y_i^*}{\sum_{i=1}^m S_{A_i B_i}(x^*)}.$$

where

$$S_{A_i B_i}(x^*) = \int_{-r_2}^{r_2} \beta_{m A_i B_i}(x^*, y) dy,$$

or

$$S_{A_i B_i}(x^*) = \int_{-r_2}^{r_2} \beta_{p A_i B_i}(x^*, y) dy \quad (i = 1, 2, \dots, m)$$

Height method:

$$\rho_{hAB}(x^*) = \frac{\sum_{i=1}^m \alpha_{A_i}(x^*) y_i^*}{\sum_{i=1}^m \alpha_{A_i}(x^*)}.$$

Since these calculations are depend on the membership functions, the subscripts A_{ij}, A_i, B_i, A , and B are put on $\alpha_m, \alpha_p, \beta_m, \beta_p, \rho_a, \rho_h$, and S .

V. ADMISSIBLE FUZZY CONTROLLER

The reasoning calculation and defuzzification are denoted as composite function through the inference procedure from 1 to 4 on the set of premise valuable $[-r_1, r_1]^n$, and depend on the set of membership function. To avoid making the denominator of the expression in the procedure 3 and defuzzification stage equal to 0, for any $\delta > 0$, and $\sigma > 0$, we consider the subset

$$\mathcal{F}_{\delta, \sigma} = \left\{ (A, B) \in \mathcal{F} : \forall x \in [-r_1, r_1]^n, \sum_{i=1}^m \alpha_{A_i}(x) \geq \sigma, \right. \\ \left. \forall i = 1, 2, \dots, m, \int_{-r_2}^{r_2} \mu_{B_i}(y) dy \geq \delta \right\}, \quad (12)$$

where α_{A_i} is $\alpha_{m A_i}$ or $\alpha_{p A_i}$. This is a slight modification of \mathcal{F} by (11). If δ and σ are taken small enough, it is possible to consider $\mathcal{F} = \mathcal{F}_{\delta, \sigma}$ for practical applications.

VI. CONTINUITY OF DEFUZZIFICATION AS FUNCTIONAL

In this section, the continuity of approximate reasoning as functional on the set of membership functions $\mathcal{F}_{\delta, \sigma}$ is shown.

A. Area Method

It is already shown that the calculations in the procedure 1 and 2 are continuous even if minimum operation or product operation [11] [12]. That is, if for each $i = 1, 2, \dots, m$,

$$\mu_{A_{ij}^k} \rightarrow \mu_{A_{ij}} \quad (i = 1, 2, \dots, m; j = 1, 2, \dots, n)$$

for $k \rightarrow \infty$ implies

$$\|\alpha_{m A_i^k} - \alpha_{m A_i}\|_{\infty} = \sup_{x \in [-r_1, r_1]^n} |\alpha_{m A_i^k}(x) - \alpha_{m A_i}(x)| \rightarrow 0$$

and

$$\|\alpha_{p A_i^k} - \alpha_{p A_i}\|_{\infty} = \sup_{x \in [-r_1, r_1]^n} |\alpha_{p A_i^k}(x) - \alpha_{p A_i}(x)| \rightarrow 0.$$

Assume that a sequence $(A^k, B^k) \subset \mathcal{F}_{\delta, \sigma}$ converges to (A, B) for the product topology if and only if, for each $i = 1, 2, \dots, m$,

$$\|\alpha_{A_i^k} - \alpha_{A_i}\|_{\infty} \rightarrow 0$$

and

$$\|\mu_{B_i^k} - \mu_{B_i}\|_{\infty} = \sup_{y \in [-r_2, r_2]} |\mu_{B_i^k}(y) - \mu_{B_i}(y)| \rightarrow 0$$

where α_{A_i} is $\alpha_{m A_i}$ or $\alpha_{p A_i}$. Then

$$\lim_{k \rightarrow \infty} \|\beta_{A_i^k B_i^k} - \beta_{A_i B_i}\|_{\infty} \\ = \lim_{k \rightarrow \infty} \sup_{x, y} |\beta_{A_i^k B_i^k}(x, y) - \beta_{A_i B_i}(x, y)| = 0, \quad (13)$$

where $\beta_{A_i B_i}$ is $\beta_{m A_i B_i}$ or $\beta_{p A_i B_i}$.

Noting that for all $i = 1, 2, \dots, m$, $\int_{-r_2}^{r_2} \mu_{B_i}(y) dy \geq \delta$ and $|\alpha_{A_i}(x)| \leq 1$ ($\forall x \in [-r_1, r_1]^n$) by (12) and the definition of membership function, we have

$$|y_i^{*k} - y_i^*| = \left| \frac{\int_{-r_2}^{r_2} y \mu_{B_i^k}(y) dy}{\int_{-r_2}^{r_2} \mu_{B_i^k}(y) dy} - \frac{\int_{-r_2}^{r_2} y \mu_{B_i}(y) dy}{\int_{-r_2}^{r_2} \mu_{B_i}(y) dy} \right| \\ \leq \frac{2r_2^3}{\delta^2} \|\beta_{A_i^k B_i^k} - \beta_{A_i B_i}\|_{\infty}. \quad (14)$$

This means that the defuzzified value y_i^* is continuous on $\mathcal{F}_{\delta, \sigma}$ by (13). It is easy to show that

$$|S_{A_i^k B_i^k}(x) - S_{A_i B_i}(x)| \leq 2r_2 \|\beta_{A_i^k B_i^k} - \beta_{A_i B_i}\|_{\infty}. \quad (15)$$

It follows from routine calculation that

$$\left| \rho_{a A^k B^k}(x) - \rho_{a AB}(x) \right| \\ = \left| \frac{\sum_{i=1}^m S_{A_i^k B_i^k}(x) y_i^{*k}}{\sum_{i=1}^m S_{A_i^k B_i^k}(x)} - \frac{\sum_{i=1}^m S_{A_i B_i}(x) y_i^*}{\sum_{i=1}^m S_{A_i B_i}(x)} \right| \\ \leq \frac{4r_2^2}{m\delta^2} \left\{ \sum_{i=1}^m |y_i^{*k} - y_i^*| + \sum_{i=1}^m |S_{A_i^k B_i^k}(x) - S_{A_i B_i}(x)| \right\}$$

by noting that $\sum_{i=1}^m S_{A_i^k B_i^k}(x) \geq m\delta$. Hence, $(A^k, B^k) \rightarrow (A, B)$ implies that

$$\|\rho_{a A^k B^k} - \rho_{a AB}\|_{\infty} = \sup_{x \in [-r_1, r_1]^n} |\rho_{a A^k B^k}(x) - \rho_{a AB}(x)| \rightarrow 0$$

for $k \rightarrow \infty$ by (14) and (15). Therefore the continuity of the area method on $\mathcal{F}_{\delta, \sigma}$ is obtained.

B. Height Method

In the same way as the area method, the following inequality is obtained:

$$\left| \rho_{h A^k B^k}(x) - \rho_{h AB}(x) \right| \\ = \left| \frac{\sum_{i=1}^m \alpha_{A_i^k}(x) y_i^{*k}}{\sum_{i=1}^m \alpha_{A_i^k}(x)} - \frac{\sum_{i=1}^m \alpha_{A_i}(x) y_i^*}{\sum_{i=1}^m \alpha_{A_i}(x)} \right| \\ \leq \frac{m}{\sigma^2} \sum_{i=1}^m |y_i^{*k} - y_i^*| + \frac{2mr_2}{\sigma^2} \sum_{i=1}^m |\alpha_{A_i^k}(x) - \alpha_{A_i}(x)|.$$

It is easy to lead that the functional ρ_h is continuous on $\mathcal{F}_{\delta, \sigma}$ from above inequality.

VII. LIPSCHITZ CONTINUITY OF DEFUZZIFICATION

In this section, Lipschitz continuity of the defuzzification as the composite function on the premise variable $[-r_1, r_1]$ is shown. And Lipschitz condition is applied to the existence of unique solution of the state equation (2).

A. Area Method

For all $i = 1, 2, \dots, m$, the following mappings α_{m, A_i} and α_{p, A_i} are Lipschitz continuous on the space of premise variables $[-r_1, r_1]^n$ [11] [12]. That is, for $x, x' \in [-r_1, r_1]^n$

$$|\alpha_{A_i}(x) - \alpha_{A_i}(x')| \leq \Delta_{\alpha_i} \|x - x'\|$$

and

$$|\beta_{A_i B_i}(x, y) - \beta_{A_i B_i}(x', y)| \leq \Delta_{\alpha_i} \|x - x'\|,$$

where Δ_{α_i} is Lipschitz constant of α_{A_i} ($i = 1, 2, \dots, m$). Then we have

$$|S_{A_i B_i}(x) - S_{A_i B_i}(x')| \leq 2r_2 \Delta_{\alpha_i} \|x - x'\|.$$

Moreover, it follows from above Lipschitz continuity of $S_{A_i B_i}$ that

$$\begin{aligned} & |\rho_{aAB}(x) - \rho_{aAB}(x')| \\ & \leq \frac{1}{m^2 \delta^2} \left\{ \left| \sum_{i=1}^m S_{A_i B_i}(x') \right| \sum_{i=1}^m |y_i^*| |S_{A_i B_i}(x) - S_{A_i B_i}(x')| \right. \\ & \quad \left. + \frac{1}{m^2 \delta^2} \left| \sum_{i=1}^m y_i^* S_{A_i B_i}(x') \right| \sum_{i=1}^m |S_{A_i B_i}(x') - S_{A_i B_i}(x)| \right\} \\ & \leq \frac{8r_2^3}{\delta^2} \sum_{i=1}^m \Delta_{\alpha_i} \|x - x'\|, \end{aligned}$$

by noting that $|y_i^*| \leq r_2$. Therefore defuzzification by area method is Lipschitz continuous on the space of premise variables $[-r_1, r_1]^n$.

B. Height Method

Since $|\alpha_{A_i}(x)| \leq 1$ for $\forall x \in [-r_1, r_1]^n$,

$$\sigma \leq \sum_{i=1}^m \alpha_{A_i}(x) \leq m.$$

Then we have, similarly

$$|\rho_{hAB}(x) - \rho_{hAB}(x')| \leq \frac{2mr_2}{\sigma^2} \sum_{i=1}^m \Delta_{\alpha_i} \|x - x'\|.$$

This inequality shows Lipschitz continuity of height method.

C. Existence of Unique Solution of The State Equation

It is easily seen that every bounded Lipschitz function $\rho : [-r_1, r_1]^n \rightarrow \mathbb{R}$ can be extended to a bounded Lipschitz function $\tilde{\rho}$ on \mathbb{R}^n without increasing its Lipschitz constant and bound. In fact, define $\tilde{\rho} : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\begin{aligned} \tilde{\rho}(x) &= \tilde{\rho}(x_1, \dots, x_n) \\ &= \begin{cases} \rho(x_1, \dots, x_n), & \text{if } x \in [-r_1, r_1]^n \\ \rho(\varepsilon(x_1)r_1, \dots, \varepsilon(x_n)r_1), & \text{if } x \notin [-r_1, r_1]^n, \end{cases} \end{aligned}$$

where

$$\varepsilon(u) = \begin{cases} 1, & \text{if } u > r_1 \\ -1, & \text{if } u < -r_1. \end{cases}$$

Let $(A, B) \in \mathcal{F}_{\delta, \sigma}$. Then it follows from Lipschitz continuity in the previous section and the fact above that the extension $\tilde{\rho}_{AB}$ of ρ_{AB} is Lipschitz continuous on \mathbb{R}^n with the same Lipschitz constant of ρ_{AB} and satisfies $\sup_{u \in \mathbb{R}^n} |\tilde{\rho}_{AB}(u)| \leq r_2$. Therefore, by proposition the state equation (3) for the feedback law $\tilde{\rho}_{AB}$ has a unique solution $x(t, x_0, \tilde{\rho}_{AB})$ with the initial condition $x(0) = x_0$ [16]. Though the extension $\tilde{\rho}_{AB}$ of ρ_{AB} is not unique in general, the solution $x(t, x_0, \tilde{\rho}_{AB})$ is uniquely determined by ρ_{AB} using the inequality (7) of (b) of proposition. Consequently, in the following the extension $\tilde{\rho}_{AB}$ is written as ρ_{AB} without confusion.

VIII. APPLICATION FOR OPTIMIZATION

In this section, using an idea and framework mentioned in the previous section, the existence of optimal control based on fuzzy rules in the admissible fuzzy controller will be established.

The performance index of this control system for the feedback law ρ (ρ_a or ρ_h) in the previous section is evaluated with the following integral performance function:

$$J = \int_{B_r} \int_0^T w(x(t, \zeta, \rho), \rho(x(t, \zeta, \rho))) dt d\zeta. \quad (16)$$

where $w : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is a positive continuous function. The following theorem guarantee the existence of a rule set which minimizes the previous function (16). In the following, ρ_{aAB} and ρ_{hAB} are written as ρ_{AB} because the difference between them does not influence the theorem.

Theorem. The mapping

$$\mathcal{F}_{\delta, \sigma} \ni (A, B) \mapsto$$

$$\int_{B_r} \int_0^T w(x(t, \zeta, \rho_{AB}), \rho_{AB}(x(t, \zeta, \rho_{AB}))) dt d\zeta$$

has a minimum value on the compact metric space $\mathcal{F}_{\delta, \sigma}$ defined by (12).

Proof. It is sufficient to prove that compactness of $\mathcal{F}_{\delta, \sigma}$ and the continuity of performance function J on $\mathcal{F}_{\delta, \sigma}$ are obtained.

For each $i = 1, 2, \dots, m$, $\mathcal{F}_{\Delta_{ij}}$ is a subset of $C[-r_1, r_1]$ which is the subspace of continuous function on $[-r_1, r_1]$. Then it is compact respect for uniform norm $\|\cdot\|_\infty$ by By the Ascoli Arzela's theorem [17]. Similarly G is a compact set. Therefore, by the Tychonoff's theorem, \mathcal{F} is compact respect for the product topology. Assume that $\{(\mathcal{A}^k, \mathcal{B}^k)\} \subset \mathcal{F}_{\delta, \sigma} \rightarrow$

$(\mathcal{A}, \mathcal{B}) \in \mathcal{F}$ for $k \rightarrow \infty$. Fix $x \in [-r_1, r_1]^n$. It is easy to show that for $i = 1, 2, \dots, m$

$$\int_{-r_2}^{r_2} \mu_{B_i}(y) dy = \lim_{k \rightarrow \infty} \int_{-r_2}^{r_2} \mu_{B_i^k}(y) dy \geq \delta$$

and

$$\sum_{i=1}^m \alpha_{A_i}(x) = \lim_{k \rightarrow \infty} \sum_{i=1}^m \alpha_{A_i^k}(x) \geq \sigma,$$

and these inequality imply $(\mathcal{A}, \mathcal{B}) \in \mathcal{F}_{\delta, \sigma}$. Therefore $\mathcal{F}_{\delta, \sigma}$ is a closed subset of \mathcal{F} and hence it is compact.

Assume that $(\mathcal{A}^k, \mathcal{B}^k) \rightarrow (\mathcal{A}, \mathcal{B})$ in $\mathcal{F}_{\delta, \sigma}$ and fix $(t, \zeta) \in [0, T] \times B_r$. Then it follows from the section VI that

$$\lim_{k \rightarrow \infty} \sup_{x \in [-r_1, r_1]^n} |\rho_{\mathcal{A}^k \mathcal{B}^k}(x) - \rho_{\mathcal{A} \mathcal{B}}(x)| = 0. \quad (17)$$

Hence, by (b) of proposition, we have

$$\lim_{k \rightarrow \infty} \|x(t, \zeta, \rho_{\mathcal{A}^k \mathcal{B}^k}) - x(t, \zeta, \rho_{\mathcal{A} \mathcal{B}})\| = 0. \quad (18)$$

Further, it follows from (17), (18) and (a) of proposition that

$$\lim_{k \rightarrow \infty} \rho_{\mathcal{A}^k \mathcal{B}^k}(x(t, \zeta, \rho_{\mathcal{A}^k \mathcal{B}^k})) = \rho_{\mathcal{A} \mathcal{B}}(x(t, \zeta, \rho_{\mathcal{A} \mathcal{B}})). \quad (19)$$

Noting that $w : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is positive and continuous, it follows from (18), (19) and the Lebesgue's dominated convergence theorem [18] that the mapping

$$(\mathcal{A}, \mathcal{B}) \in \mathcal{F}_{\delta, \sigma} \mapsto$$

$$\int_{B_r} \int_0^T w(x(t, \zeta, \rho_{\mathcal{A} \mathcal{B}}), \rho_{\mathcal{A} \mathcal{B}}(x(t, \zeta, \rho_{\mathcal{A} \mathcal{B}}))) dt d\zeta$$

is continuous on the compact metric space $\mathcal{F}_{\delta, \sigma}$. Thus it has a minimum (maximum) value on $\mathcal{F}_{\delta, \sigma}$, and the proof is complete.

IX. CONCLUSION

In this paper, we analyzed the continuity of the defuzzification and proved that there exists an optimal feedback control law in a nonlinear fuzzy feedback control system, in which the feedback laws are determined by IF-THEN type fuzzy rules. Actually the height method is particular case of the area method. In the case that some conditions are added to the set of membership functions, they are the same. To select the proper defuzzification method, it is necessary to analyze the property of fuzzy approximate reasoning and the simulation of fuzzy control.

It is recognized that in various applications it could be a useful tool in analyzing the convergence of fuzzy control rules modified recursively.

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