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On Generalizing Rough Set Theory via using a Filter

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***Abstract*—**The theory of rough sets is generalized by using a filter. The filter is induced by binary relations and it is used to generalize the basic rough set concepts. The knowledge representations and processing of binary relations in the style of rough set theory are investigated.

***Keywords*—**Rough set, fuzzy set, membership function, knowledge representation and processing, information theory

I. Introduction

IN 1982, Pawlak has introduced the rough set theory [7], which has emerged as another major mathematical tool, for modelling the vagueness present in the human classification mechanism. This concept is fundamental for the examination of granularity in the knowledge [8,13,14]. It is a concept that has many applications in machine learning, pattern recognition, decision support systems, expert systems, data analysis, and data mining, among others.

The theory of rough sets can be developed by the constructive and algebraic methods [15,16,21,22]. The constructive methods define rough set approximation operators using equivalence relations or their induced partitions and subsystems; the algebraic methods treat approximations of operators as abstract operators. In this paper, we introduce a new approach for computing rough membership function using a topological filter instead of the equivalence relation in Pawlak's approach.

II. Preliminaries

The partition characterizes a topological space, called approximation space K = (U,R), where U is a set called the universe and R is an equivalence relation [4,9]. The equivalence classes of R are also known as the granules, elementary sets or blocks; we will use R(x)⊆U to denote the equivalence class containing x∈U.

Definition 2.1. Formal definitions of approximations and the boundary region are as follows[7]:

- R-lower approximation of X

$$R_*(X)=\{x\in U: R(x)\subseteq X\}.$$

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The lower approximation of a set X with respect to R is the set of all objects, which can be for certain classified as X with respect to R (are certainly X with respect to R))

R-upper approximation of X

$$R^*(X)=\{x\in U: R(x)\cap X\neq\varnothing\}$$

The upper approximation of a set X with respect to R is the set of all objects which can be possibly classified as X with respect to R (are possibly X in view of R).

R-boundary region of X

$$BN_R(X)=R^*(X)-R_*(X).$$

The boundary region of a set X with respect to R is the set of all objects, which can be classified neither as X nor as not-X with respect to R.

Definition 2.2. Positive and negative regions are also defined:

i. $POS_R(X)=R_*(X)$

ii. $NEG_R(X)=U-R^*(X)$.

These notions can be also expressed by rough membership functions [10,9], namely,

$$\mu_X^R: U\to<0,1>,$$

where

$$\mu_X^R(x)=\frac{|X\cap R(x)|}{|R(x)|}$$

and $|X|$ denotes the cardinality of X.

Different values define boundary ($0<\mu_X^R(x)<1$), positive ($\mu_X^R(x)=1$) and negative ($\mu_X^R(x)=0$) regions. The membership function is a kind of the probability and its value can be interpreted as a degree of certainty to which x belongs to X. Also the rough membership function can be used to define approximations:

$$R_*(X)=\{x\in U:\mu_X^R(x)=1\},$$

$$R^*(X)=\{x\in U:\mu_X^R(x)>0\}.$$

Now we are ready to give the definition of rough sets.

Set X is *crisp* (exact with respect to R), if the boundary region of X is empty.

Set X is *rough* (inexact with respect to R), if the boundary region of X is nonempty.

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Definition 2.3. A filter F on U is then a subset of P(U) with the following properties:

1. U is in F. (F is non-empty)
2. The empty set is not in F. (F is proper)
3. If *A* and *B* are in *F*, then so is their intersection. (*F is closed under finite meets*)
4. If *A* is in *F* and *A* is a subset of *B*, then *B* is in *F*, for all subsets *B* of *U*. (*F is an upper set*)

The first three properties imply that a filter has the finite intersection property[2].

Definition 2.4. A filter base is a subset β of P(*U*) with the following properties

1. The intersection of any two sets of β contains a set of β
2. β is non-empty and the empty set is not in β

A filter base β can be turned into a filter by including all sets of P(*U*) which contain a set of β [2].

III. TOPOLOGICAL FILTER AND ROUGH MEMBERSHIP FUNCTION

The theory of rough sets can be generalized in several directions. Within the set-theoretic framework, generalizations of the element based definition can be obtained by using non-equivalence binary relations [11,17,18,24,26], generalizations of the granule based definition can be obtained by using coverings [11, 12, 19, 20, 27], and generalizations of subsystem based definition can be obtained by using other subsystems [16, 25]. By the fact that the system $(2^U, ^c, \cap, \cup)$ is a Boolean algebra, one can generalize rough set theory using other algebraic systems such as Boolean algebras, lattices, and posets [1, 18, 16]. Subsystem based definition and algebraic methods are useful for such generalizations [23].

We will use a filter; in other words, the ''approximation space'' is a filter. Original rough membership function is defined using the equivalence of classes. It was extended to a topological space by using the topological base in [3]. We will extend it to topological filters. If $\Im$ is a fitler on a finite set U and β is base of $\Im$ then the rough membership function is

$$\mu_X^{\Im}(x)=\frac{|\{\cap B_x\}\cap X|}{|\cap B_x|},\quad B_x\in\beta,\ x\in U,$$

where B_x is any member of β containing x. Let U be a finite universal set and X be any subset of U. Then the lower and upper approximation of X can be explaning by the membership function of X. One can define the following four basic classes of rough sets, i.e., four categories of vagueness:

a) $\beta_*(X)\neq\varnothing$ and $\beta^*(X)\neq U$, iff *X* is *roughly β-definable*,

b) $\beta_*(X)=\varnothing$ and $\beta^*(X)\neq U$, iff *X* is *internally β-indefinable*,

c) $\beta_*(X)\neq\varnothing$ and $\beta^*(X)=U$, iff *X* is *externally β-definable*,

d) $\beta_*(X)=\varnothing$ and $\beta^*(X)=U$, iff *X* is *totally β-indefinable*.

The following example illustrates the above definition.

Let U = {a, b, c, d, e}, β={{a, b}, {a, b, d}, {a, b, e}, {a, b, c, d}, {a, b, c, e}, {a, b, d, e}}, X={b, c, e}, we get:

$$\mu_X^{\Im}(a)=\frac{|\{a,b\}\cap\{b,c,e\}|}{|\{a,b\}|}=\frac{1}{2},$$

$$\mu_X^{\Im}(b)=\frac{1}{2},\ \mu_X^{\Im}(c)=\frac{2}{3},\ \mu_X^{\Im}(d)=\frac{1}{3},\ \mu_X^{\Im}(e)=\frac{2}{3}\ ;$$

and

$$\beta_*(X)=\{x\in U:\mu_X^{\beta}(x)=1\}=\varnothing,$$

$$\beta^*(X)=\{x\in U:\mu_X^{\beta}(x)>0\}=\{a,b,c,d,e\},$$

then *X* is *totally B-indefinable.*

Also rough membership functions allow us to express fuzzy theory in topological spaces: Let $X\subseteq U$ be a subset, we define a fuzzy set by using the fitler rough membership function:

$$\underset{\sim}{X}=\{(x,\mu_X^{\Im}(x)):\forall x\in U\}.$$

From the above example, if X={b, c, e} then $\underset{\sim}{X}=\{(a,1/2),(b,1/2),(c,2/3),(d,1/3),(e,2/3)\}$

Proposition 3.1. Let U denotes a finite (universal) set and X⊂U. If X is internally β1-indefinable and β1⊂β2 then X is internally β2-indefinable, where β1 and β2 are two filter bases.

Proof. Since $\beta_{2*}(X)\subset\beta_{1*}(X)$ and $\beta_{1*}(X)=\varnothing$ then $\beta_{2*}(X)=\varnothing$, also

Since $\beta_{2*}(X)\subset\beta_{1*}(X)$ and $\beta_{1*}(X)\neq U$ then $\beta_{2*}(X)\neq U$,and X is internally β_2-indefinable.

Proposition 3.2. Let U denotes a finite (universal) set and X⊂U. If X is externally β2-definable and β1⊂β2 then X is externally β1-definable, where β1 and β2 are two filter bases.

Proof. Since $\beta_{2*}(X)\subset\beta_{1*}(X)$ and $\beta_{2*}(X)\neq\varnothing$ then $\beta_{1*}(X)\neq\varnothing$, also

Since $\beta_{2*}(X)\subset\beta_{1*}(X)$ and $\beta_{2*}(X)=U$ then $\beta_{1*}(X)=U$,and X is externally β_1-definable.

IV. ROUGH SET THEORY IN THE FILTER OF BINARY RELATION

Lin has introduced[5] the formalism of neighborhood system to handle such general situations. In [3] the topology generated from the binary relation R is considered. If U is a finite universe and R is a binary relation on U, then we define a right neighborhood

xR={y : xRy}

We should note that xR is a right neighborhood of x, but xR

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is not necessary a right neighborhood of any element in xR. In fact, the set of all elements, each of which has xR as its right neighborhood, is called the center of xR. The collections of all centers form a partition of U; see [6] for details.

We will not consider a right neighborhood system, we will consider the filter generated by right neighborhoods which has a nonempty finite intersection. To construct the filter, we consider the family S = {xR: x∈U} of right neighborhoods as a subbase. Let the induced filter be $\Im$. The family S as the subbase of $\Im$ will be denoted by SR = {xR: x∈U }, and we write Sx = {G ∈SR: x ∈G}.

Since all finite intersections of members of a subbase form a base, the notion of fitler rough membership functions can be expressed by subbase:

$$\mu_X^{\Im}(x)=\frac{|\{\cap S_x\}\cap X|}{|\cap S_x|},\quad S_x\in\Im,\ x\in U$$

Note that this rough membership is very different from rough set theory or Lin's rough membership function of a right neighborhood or [3]. In Lin's case instead of Sx, he will use xR, which is unique. In [3], however, S_R is used to topological subbased; furthermore S_R does not need to be closed under finite intersection.

Example 4.1.

Let U={a, b, c, d, e}, aR=bR=cR={b, c, d}, dR={a, b, c}, eR={a, b, c, e} Then

S={{b, c, d}, {a, b, c}, {a, b, c, e}} and β={{b, c}, {b, c, d}, {a, b, c}, {a, b, c, e}} →

$\Im$ ={{b, c}, {b, c, d}, {a, b, c}, {b, c, e}, {a, b, c, e}, {a, b, c, d}, {b, c, d, e}, {a, b, c, d, e}}

Let X={b, c, d}

$$\mu_X^{\Im}(a)=\frac{|\{a,b,c\}\cap\{b,c,d\}|}{|\{a,b,c\}|}=\frac{2}{3},$$

$$\mu_X^{\Im}(b)=1,\ \mu_X^{\Im}(c)=1,\ \mu_X^{\Im}(d)=1,\ \mu_X^{\Im}(e)=\frac{2}{3}$$

Then $\underset{\sim}{X}=\{(a,2/3),(b,1),(c,1),(d,1),(e,2/3)\}$

From the rough membership function, we get:

$$R_*(X)=\{b,c,d\},\ R^*(X)=\{a,b,c,d,e\},$$
$$NEG_R(X)=\varnothing,\ BN_R(X)=\{a,e\}.$$

V. Granular Structure in the Filter of Binary Relations

The purpose of this section is to investigate the knowledge representations and processing of binary relations in the style of rough set theory. Let us consider the pair, (U,B), where B = {R1,R2, . . . ,Rn} is a family of general binary relations right neighborhood of each of whom has nonempty finite intersection on the universe U. When B is a family of equivalence relations, Pawlak call it knowledge base and Lin call the general case binary knowledge base in [6]. As the term ''knowledge base'' often means something else, Lin begin to use the generic name granular structure [6,5]. We will use the knowledge structure and the granular structure interchangeably.

Next, we will consider the filter for each binary relation; we will call it the filter of the binary relation (FRB). We denote the base β_R that is generated by the binary relation R. Note that two distinct binary relations R_1 and R_2 may generate the same fiter as shown in the following example: Let U = {a, b,c,d}, R_1 and R_2 are distinct binary relations, where,

R_1={(a,a), (a,b), (a,c), (b,b), (b,a), (c,a), (c,b), (a,d)}
R_2={(a,a), (a,b), (b,b), (b,a), (b,c), (c,a), (c,b), (a,d)}
Their (right) neighborhood systems are: (as subbases)
aR_1={a, b, c}, bR_1={a, b}, cR_1={a, b, d},
aR_2 ={a, b}, bR_2 ={a, b, c}, cR_2={a, b, d}

These two subbases generated the same base S_{R_1} ={{a,b}, {a, b, c}, {a, b, d}}= S_{R_2} hence the same filter $\Im_{R_1}=\Im_{R_2}$

In [2] the notion of reducts to TSB (the topological space of the binary relation) was generalized. Next, we will generalize the notion of reducts to FRB, the filter of binary relations.

Definition 5.1. Let P⊆ B be a subset of B, r ∈ P, where B be a class of binary relations right neighborhood of each of whom has nonempty finite intersection. r is said to be superfluous binary relation in P if:

$\beta_P=\beta_{(P-\{r\})}$
The set M is called a minimal reduct of P iff:
i. $\beta_M=\beta_{(P)}$
ii. $\beta_M\neq\beta_{(P-\{r\})}$, ∀r∈M.

The following example illustrates the notion given above.

Example 5.2.

Let U= {a, b, c, d, e} and three subbases S_r={{d}, {d,e}, {a,d,e}, {a,c,d,e}}, S_p={{d}, {a,d,e}, {b,c,d }} S_q={{d}, {a,d}, {b,c,d}, {a,c,d,e}} Then we have a joint subbase S_B={{d}, {a,d}, {d,e}, {a,d,e}, {b,c,d}, {a,c,d,e}}.

The base is β_B={{d}, {a,d}, {d,e}, {c,d}, {a,d,e}, {b,c,d}, {a,c,d,e}}. Next consider

$S_{(B-r)}$={{d}, {a,d}, {a,d,e}, {b,c,d}, {a,c,d,e}},
$B_{(B-r)}$={{d}, {a,d}, {c,d}, {a,d,e}, {b,c,d}, {a,c,d,e}},

$S_{(B-p)}$={ {d}, {a,d}, {d,e}, {a,d,e}, {b,c,d}, {a,c,d,e}},
$B_{(B-p)}$={{d}, {a,d}, {d,e}, {c,d}, {a,d,e}, {b,c,d}, {a,c,d,e}} =β_B,

$S_{(B-q)}$={{d}, {d,e}, {a,d,e}, {b,c,d}, {a,c,d,e}},
$B_{(B-q)}$={{d}, {d,e}, {c,d}, {a,d,e}, {b,c,d}, {a,c,d,e}}.

So we find that p is only superfluous relation in B, and we have

RED(B)= {r,q}, CORE(B)={r,q}.

VI. Conclusion

In this study, we generalize the rough set theory via using a

filter. We suppose that such generalized rough set theory will be useful in various aria. Our theory connects rough sets, filters, fuzzy sets, and neighborhood systems. We belive that other topological concepts can use to generalize rough sets and these concepts are associated in the frameworks of topological spaces.

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