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Dynamic Stability of Axially Moving Viscoelastic Plates under Non-Uniform In-Plane Edge Excitations

T. H. Young, S. J. Huang, Y. S. Chiu

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***Abstract*—**This paper investigates the parametric stability of an axially moving web subjected to non-uniform in-plane edge excitations on two opposite, simply-supported edges. The web is modeled as a viscoelastic plate whose constitutive relation obeys the Kelvin-Voigt model, and the in-plane edge excitations are expressed as the sum of a static tension and a periodical perturbation. Due to the in-plane edge excitations, the moving plate may bring about parametric instability under certain situations. First, the in-plane stresses of the plate due to the non-uniform edge excitations are determined by solving the in-plane forced vibration problem. Then, the dependence on the spatial coordinates in the equation of transverse motion is eliminated by the generalized Galerkin method, which results in a set of discretized system equations in time. Finally, the method of multiple scales is utilized to solve the set of system equations analytically if the periodical perturbation of the in-plane edge excitations is much smaller as compared with the static tension of the plate, from which the stability boundaries of the moving plate are obtained. Numerical results reveal that only combination resonances of the summed-type appear under the in-plane edge excitations considered in this work.

***Keywords*—**Axially moving viscoelastic plate, in-plane periodic excitation, non-uniformly distributed edge tension, dynamic stability.

I. Introduction

AXIALLY moving materials are frequently encountered in industrial applications, such as magnetic tapes, transmission belts, paper webs, plastic sheets, band saw blades, etc.. Therefore, hundreds of references pertaining to the dynamic behaviors of axially moving materials can be found in the literature. Depending on the aspect ratio of the axially moving webs between two adjacent supports, they can be regarded as one-dimensional structures or two-dimensional structures. Furthermore, according to the flexural rigidity of the axially moving webs, they can be modeled as strings [1], [2] or beams [3] for webs with large aspect ratios and as membranes [4], [5] or plates [6], [7] for webs with small aspect ratios.

Earlier researchers deal primarily with the determination of the critical speed of the materials moving axially with constant speeds and subjected to static, uniformly distributed tensions. However, recent researches focus primarily on the dynamic response and stability of the axially moving materials under various complicated effects, such as nonconstant moving speeds, time-dependent and/or nonuniformly distributed tensions, etc. In reality, the moving speed and the applied tension of axially moving webs are generally time-dependent. Due to the time-dependent moving speed or the time-dependent tension, an axially moving plate may experience parametric instability before the moving speed of the plate exceeds the first critical speed.

Chen and Yang [8] investigated the nonlinear response at principal parametric resonances for viscoelastic beams moving axially with a pulsating speed. The pulsating speed of the moving beam is assumed to be the sum of a constant speed and a small, harmonically varying perturbation, and the material property of the viscoelastic beam obeys the Kelvin model. Marynowski [9] studied the nonlinear vibrations of axially moving viscoelastic beams with a constant speed and subjected to a time-dependent tension. The time-dependent tension is assumed to a small, harmonically varying perturbation superimposed on a static tension, and the internal damping of the viscoelastic beam is introduced by the Kelvin-Voigt model. Both references deal with axially moving webs having large aspect ratio. Tang and Chen [10] analyzed the parametric instability of axially moving viscoelastic plates with a time-dependent speed. The time-dependent speed of the moving plate is assumed to be the sum of a constant speed and a small, harmonically varying perturbation, and the tension is assumed to be uniformly distributed on two opposite edges. Liu [11] investigated the dynamic stability of an axially moving web which translates with periodically varying speeds and is subjected to partially distributed tensions on two opposite simply-supported edges. The web, having a small aspect ratio, is modeled as a viscoelastic rectangular plate which obeys the Kelvin-Voigt model. The moving speed of the plate is expressed as the sum of a constant speed and a small, periodical perturbation. Dynamic instability of a moving plate subjected to uniformly-distributed parametric in-plane forces was analyzed by [12]. The extended Galerkin method is used to discretize the differential equation of motion, and the harmonic balance method is employed for investigating the instability regions of the moving plate.

This paper investigates the parametric stability of an axially moving web subjected to nonuniform in-plane edge excitations on two opposite, simply-supported edges. The web is modeled as a viscoelastic plate whose constitutive relation obeys the Kelvin-Voigt model, and the in-plane edge excitations are expressed as the sum of a static tension and a periodical perturbation. Due to the in-plane edge excitations, the moving plate may bring about parametric instability under certain situation. First, the in-plane stresses of the plate due to the nonuniform edge excitations are determined by solving the in-plane forced vibration problem. Then, the dependence on the spatial coordinates in the equation of transverse motion is eliminated by the generalized Galerkin method, which results

T. H. Young is with the National Taiwan University of Science and Technology, Taiwan, (e-mail: thyoung@mail.ntust.edu.tw).

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in a set of discretized system equations in time. Finally, the method of multiple scales is utilized to solve the set of system equations analytically if the periodical perturbation of the in-plane edge excitations is much smaller as compared with the static tension of the plate, from which the stability boundaries of the moving plate are obtained.

II. Equations of Motion

Consider a rectangular plate of dimension $a \times b$ moving in the x-direction with a constant speed V. The plate is simply supported on two opposite edges $x = 0$ and a, and is free on the other two edges $y = 0$ and b. The plate is subjected to a periodic excitation $f(y,t)$ arbitrarily distributed on two simply supported edges, as shown in Fig. 1. The plate possesses an internal damping of which the constitutive relation obeys the Kelvin-Voigt model, i.e.

$$\sigma = E\left(\epsilon + c\frac{\partial \epsilon}{\partial t}\right), \tag{1}$$

where σ and ϵ are the stress and strain of the plate, respectively, and t is the temporal variable. E and c are Young's modulus and internal damping coefficient of the plate, respectively.

The equations of motion for such a plate are written as:

$$\rho h\left(\frac{\partial^2 u}{\partial t^2} + 2V\frac{\partial^2 u}{\partial t \partial x} + V^2\frac{\partial^2 u}{\partial x^2}\right) - \frac{Eh}{1-\nu^2}\left(\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 v}{\partial x \partial y}\right)$$
$$-\frac{Eh}{2(1+\nu)}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) - \frac{Ehc}{1-\nu^2}\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 v}{\partial x \partial y}\right)$$
$$-\frac{Ehc}{2(1+\nu)}\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) = f(y,t)[\delta(x-a) - \delta(x)], \tag{2a}$$

$$\rho h\left(\frac{\partial^2 v}{\partial t^2} + 2V\frac{\partial^2 v}{\partial t \partial x} + V^2\frac{\partial^2 v}{\partial x^2}\right) - \frac{Eh}{1-\nu^2}\left(\frac{\partial^2 v}{\partial y^2} + \nu\frac{\partial^2 u}{\partial x \partial y}\right)$$
$$-\frac{Eh}{2(1+\nu)}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}\right) - \frac{Ehc}{1-\nu^2}\frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial y^2} + \nu\frac{\partial^2 u}{\partial x \partial y}\right)$$
$$-\frac{Ehc}{2(1+\nu)}\frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}\right) = 0, \tag{2b}$$

$$\rho h\left(\frac{\partial^2 w}{\partial t^2} + 2V\frac{\partial^2 w}{\partial t \partial x} + V^2\frac{\partial^2 w}{\partial x^2}\right) + \left(1 + c\frac{\partial}{\partial t}\right)D\nabla^4 w$$
$$= \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_{yy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial x}\left(N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x}\right), \tag{2c}$$

where u and v are the in-plane displacements in the x- and y-directions, respectively, and w is the transverse displacement of the moving plate. ρ and h are the mass density and the thickness of the plate, respectively. D and ν are the flexural rigidity and Poisson's ratio of the plate, respectively. $\delta(\cdot)$ is a Dirac delta function, and ∇^4 is a Laplacian operator in Cartesian coordinates. N_{xx}, N_{yy}, and N_{xy} are the in-plane stress resultants of the plate, and are related to the in-plane displacements by:

$$N_{xx} = \frac{Eh}{1-\nu^2}\left[\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right) + c\left(\frac{\partial^2 u}{\partial t \partial x} + \nu\frac{\partial^2 v}{\partial t \partial y}\right)\right], \tag{3a}$$

$$N_{yy} = \frac{Eh}{1-\nu^2}\left[\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right) + c\left(\frac{\partial^2 v}{\partial t \partial y} + \nu\frac{\partial^2 u}{\partial t \partial x}\right)\right], \tag{3b}$$

$$N_{xy} = \frac{Eh}{2(1+\nu)}\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + c\left(\frac{\partial^2 u}{\partial t \partial y} + \nu\frac{\partial^2 v}{\partial t \partial x}\right)\right], \tag{3c}$$

Equations (2a) and (2b) are reduced to those obtained by [13] if the internal damping force terms are deleted, and (2c) is the same as that appeared in literature [6], [7]. The boundary conditions of the moving plate are as,

$$\text{at } x = 0 \text{ and } a,\ N_{xx} = 0 \text{ and } v = 0;\ \ w = 0 \text{ and } \frac{\partial^2 w}{\partial x^2} = 0, \tag{4a}$$

$$\text{at } y = 0 \text{ and } b,\ N_{yy} = 0 \text{ and } N_{xy} = 0;\ \frac{\partial^2 w}{\partial y^2} = 0 \text{ and}$$
$$\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial y \partial x^2} = 0, \tag{4b}$$

To solve the out-of-plane equation, (2c), one has to determine the in-plane response u and v from (2a) and (2b) first, from which the in-plane stress resultants are obtained. Equations (2a) and (2b) are coupled and are so complicated that the exact solution of them is not feasible. To obtain an approximate solution in the finite dimensional space, discretization of the system equations is carried out. Due to the boundary conditions of the problem, it is very difficult to find comparison functions, the ones that satisfy both the geometric and natural boundary conditions, as trial functions. Therefore, the extended Ritz method, which admits admissible functions that satisfy the geometric boundary conditions only as trial functions, is adopted in conjunction with the extended Hamilton principle,

$$\int_{t_1}^{t_2}(\delta T - \delta U + \delta W_k + \delta W_d + \delta M_v)dt = 0, \tag{5}$$

where δT and δU are variations of the kinetic and strain energies of the moving plate, respective. δW_k and δW_d are the virtual works done by the edge excitations and the internal damping force, respectively. δM_v is the virtual momentum transport across the system boundaries [13].

Assume the displacement components u and v to be of the forms;

$$u(x,y,t) = \sum_{m=0}^{M}\sum_{n=0}^{N} A_{mn}(t)\left(\frac{x}{a}\right)^m\left(\frac{y}{b}\right)^n, \tag{6a}$$

$$v(x,y,t) = \sum_{m=0}^{M}\sum_{n=0}^{N} B_{mn}(t)\sin\frac{(m+1)\pi x}{a}\left(\frac{y}{b}\right)^n, \tag{6b}$$

where $A_{mn}(t)$ and $B_{mn}(t)$ are undetermined functions. The assumed displacements satisfy the only geometric boundary condition v = 0 at the edge x = 0. If the periodic edge excitation is expended into a Fourier series as:

$$f(y,t) = \sum_{j=0}^{J}(f_{cj}(y)\cos j\Omega t + f_{sj}(y)\sin j\Omega t), \tag{7}$$

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where f_{cj} and f_{sj} are Fourier coefficients, and Ω is the fundamental frequency of the series, on substituting the assumed displacement components u and v into (5) yields;

$$[M_i]\ddot{\boldsymbol{U}} + (V[G_i] + c[K_{ie}])\dot{\boldsymbol{U}} + \left([K_{ie}] + V^2[K_{ig}]\right)\boldsymbol{U} = \sum_{j=0}^{J}(\boldsymbol{F}_{cj}\cos j\Omega t + \boldsymbol{F}_{sj}\sin j\Omega t), \quad (8)$$

where an overdot denotes a differentiation with respect to t. The mass matrix $[M_i]$ and the elastic stiffness matrix $[K_{ie}]$ are symmetric, while the moving-speed dependent matrix $[G_i]$ and the geometric stiffness matrix $[K_{ig}]$ are asymmetric. F_{cj} and F_{sj} are force vectors, and the response vector U is a column matrix formed by all $A_{mn}(t)$ and $B_{mn}(t)$. The close-form solutions for $A_{mn}(t)$ and $B_{mn}(t)$ can be calculated from (8), and the in-plane response u and v may then be determined by (6a) and (6b), from which the in-plane stress resultants are obtained by (3a)–(3c).

Once the in-plane stress resultants are obtained, we are readily to solve the out-of-plane equation, (2c). Because the equation is a fourth order partial differential equation with variable coefficients, it is unable to be solved directly. Therefore, dependence on spatial coordinates is eliminated first, and the equation is transformed into a set of ordinary differential equations. Since expressions for in-plane stress resultants are very messy, it is impossible to find the eigensolution for (2c). Therefore, approximate methods are utilized again to obtain discretized system equations for the equation. In this work, the extended Galerkin method with simpler trial functions is adopted.

Assume the transverse displacement was:

$$w = \sum_{k=1}^{K}\sum_{l=0}^{L} C_{kl}(t)\sin\left(\frac{k\pi}{a}x\right)\left(\frac{y}{b}\right)^l \quad (9)$$

where $C_{kl}(t)$ are undetermined functions. The assumed displacement satisfies the boundary conditions at both simply supported edges $x = 0$ and a, but it does not satisfy the boundary conditions at both free edges $y = 0$ and b. On substituting the assumed displacement into the out-of-plane equation and the boundary conditions yields certain residues. The extended Galerkin method requires the overall residue weighted by the trial functions vanish, i.e.

$$\int_0^b\int_0^a \gamma\sin\left(\frac{k\pi}{a}x\right)\left(\frac{y}{b}\right)^l dxdy + \int_0^a V_y\sin\left(\frac{k\pi}{a}x\right)\left(\frac{y}{b}\right)^l\Big]_{y=0,b}dx - \int_0^a M_{yy}\sin\left(\frac{k\pi}{a}x\right)\left(\frac{d}{dy}\left(\frac{y}{b}\right)^l\right)\Big]_{y=0,b}dx = 0, \quad (10)$$

where γ, V_y and M_{yy} are residues of the out-of-plane equation, transverse shear resultants and bending moment resultants along the free edges, respectively. Substituting the assumed displacement w into (10) and nondimensionalizing all parameters yield,

$$[M_o]\boldsymbol{W}'' + \left(\tilde{V}[G_o] + \tilde{c}[K_{oe}]\right)\boldsymbol{W}' + \left([K_{oe}] + \tilde{V}^2[K_{og}] + \tilde{f}_0[F_{c0}]\right)\boldsymbol{W} = -\tilde{f}_1\sum_{j=1}^{J}\left([F_{cj}]cosj\tilde{\Omega}\tau + [F_{sj}]\sin j\tilde{\Omega}\tau\right)\boldsymbol{W}, \quad (11)$$

where the nondimensional parameters:

$$\tilde{V} = V\sqrt{\frac{\rho hb^2}{D}},\ \tilde{c} = c\sqrt{\frac{D}{\rho hb^4}},\ \tilde{f}_i = \frac{f_i b^2}{D},\ \tilde{\Omega} = \sqrt{\frac{D}{\rho hb^4}}\Omega \text{ and } \tau = t\sqrt{\frac{D}{\rho hb^4}}, \quad (12)$$

in which f_0 and f_1 are representative quantities of static and dynamic components of the periodic edge excitations, respectively, and a prime denotes a differentiation with respect to τ. The mass matrix $[M_o]$, the elastic stiffness matrix $[K_{oe}]$, the geometric stiffness matrix $[K_{og}]$ and the force matrix $[F_{co}]$ are symmetric, while the gyroscopic matrix $[G_o]$ is skew-symmetric. The displacement vector W is a column matrix formed by all undetermined functions $C_{kl}(\tau)$.

Equation (11) is a set of simultaneous second-order differential equations with periodic coefficients, of which an analytic solution is still hard to find. To improve the solvability, a modal analysis suitable for gyroscopic systems is conducted to partially uncouple terms on the left-hand side of the equation. First define a state vector $q = [W'\ W]^T$, and rewrite (11) as

$$\begin{bmatrix}[M_o] & [0]\\ [0] & [K_o]\end{bmatrix}\boldsymbol{q}' + \begin{bmatrix}\tilde{V}[G_o] & [K_o]\\ -[K_o] & [0]\end{bmatrix}\boldsymbol{q} = -\tilde{c}\begin{bmatrix}[K_{oe}] & [0]\\ [0] & [0]\end{bmatrix}\boldsymbol{q} - \tilde{f}_1\sum_{j=1}^{J}\begin{bmatrix}[0] & \left([F_{cj}]\cos j\tilde{\Omega}\tau + [F_{sj}]\sin j\tilde{\Omega}\tau\right)\\ [0] & [0]\end{bmatrix}\boldsymbol{q}, \quad (13)$$

where $[K_o] = [K_{oe}] + \tilde{V}^2[K_{og}] + \tilde{f}_0[F_{c0}]$. Next consider the eigenvalue problem of the corresponding free, undamped gyroscopic system,

$$s\begin{bmatrix}[M_o] & [0]\\ [0] & [K_o]\end{bmatrix}\boldsymbol{Z} + \begin{bmatrix}\tilde{V}[G_o] & [K_o]\\ -[K_o] & [0]\end{bmatrix}\boldsymbol{Z} = \boldsymbol{0}, \quad (14)$$

The eigenvalue of the above gyroscopic system s appears in pure imaginary pairs, $s = \pm i\omega_n$, n=1, 2, …, I, where $I = K \times L$ is the total degrees of freedom of the discretized system, and the corresponding eigenvector Z appears in complex conjugate pairs, $Z = X_n \pm iY_n$. The modal matrix [P] of the discretized system is constituted by the real and the imaginary parts of the normalized eigenvectors, $[P] = [X_1, Y_1, X_2, Y_2, \ldots X_I, Y_I]$. By the orthogonality of the eigenvectors, the matrices in (14) can be diagonalized by the modal matrix, i.e.

$$[P]^T\begin{bmatrix}[M_o] & [0]\\ [0] & [K_o]\end{bmatrix}[P] = [I] \text{ and } [P]^T\begin{bmatrix}\tilde{V}[G_o] & [K_o]\\ -[K_o] & [0]\end{bmatrix}[P] = [\Lambda], \quad (15)$$

where $[P]^T$ is the transpose of [P], [I] is an identity matrix, and $[\Lambda]$ is a block-diagonal matrix of the form

$$[\Lambda] = \text{diag.}\begin{bmatrix}0 & -\omega_r\\ \omega_r & 0\end{bmatrix}. \quad (16)$$

Then, introduce a linear transformation $q = [P]\zeta$, Upon substituting this transformation into (13) and premultiplying $[P]^T$ to the equation yields

$$\boldsymbol{\zeta}' + [\Lambda]\boldsymbol{\zeta} = -\tilde{c}[C]\boldsymbol{\zeta} - \epsilon\sum_{j=1}^{J}([F_{cj}^*]\cos\ j\tilde{\Omega}\tau + [F_{sj}^*]\sin\ j\tilde{\Omega}\tau)\boldsymbol{\zeta}, \quad (17)$$

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where the matrices $[C] = [P]^T \begin{bmatrix} [K_{oe}] & [0] \\ [0] & [0] \end{bmatrix} [P]$, and $[F^*_{cj,sj}] = \tilde{f}_0 [P]^T [F_{cj,sj}][P]$; $\epsilon = \tilde{f}_1/\tilde{f}_0$ is a small quantity.

Note that the matrix [C] is again symmetric since $[K_{oe}]$ is symmetric. The terms on the left-hand side of the above equation are uncoupled in a block sense; however, those on the right-hand side are still coupled. To match the form of the matrix [Λ], matrices on the right-hand side of the equation are partitioned into I × I pieces of 2 × 2 blocks. Therefore, the above equation can be rewritten as

$$\xi_n' - \omega_n \eta_n = -\tilde{c} \sum_{m=1}^{I} (c_{nm}^{11} \xi_m + c_{nm}^{12} \eta_m) - \epsilon \sum_{j=1}^{J} \sum_{m=1}^{I} [(g_{jnm}^{11} \xi_m + g_{jnm}^{12} \eta_m) cos\, j\tilde{\Omega}\tau + (h_{jnm}^{11} \xi_m + h_{jnm}^{12} \eta_m) sin\, j\tilde{\Omega}\tau], \quad (18a)$$

$$\eta_n' + \omega_n \xi_n = -\tilde{c} \sum_{m=1}^{I} (c_{nm}^{21} \xi_m + c_{nm}^{22} \eta_m) - \epsilon \sum_{j=1}^{J} \sum_{m=1}^{I} [(g_{jnm}^{21} \xi_m + g_{jnm}^{22} \eta_m) cos\, j\tilde{\Omega}\tau + (h_{jnm}^{21} \xi_m + h_{jnm}^{22} \eta_m) \sin j\tilde{\Omega}\tau], \quad n=1, 2, \ldots, I \quad (18b)$$

where c_{nm}^{rs}, g_{jnm}^{rs} and h_{jnm}^{rs} are the r-sth entries of the n-mth block of the matrices [C], $[F^*_{cj}]$ and $[F^*_{sj}]$, respectively. ξ_n and η_n are the (2n-1)th and the 2nth entries of ζ. Although partially uncoupled, the above equation is still very hard to solve. However, if terms on the right-hand side of the equation are small in certain sense as compared with those on the left-hand side, perturbation method can be used to find an analytic solution.

III. Perturbation Analysis

In this work, the method of multiple scales is utilized to find an analytical solution of (18) if terms on the right-hand side of the equation are small in certain sense. By introducing new temporal variables $T_j = \epsilon^j \tau$, j=1, 2. …. It follows that the derivatives with respective to τ become expansions in terms of partial derivatives with respective to T_j [14], i.e., $d/d\tau = D_0 + \epsilon D_1 + \cdots$, where $D_j = \partial/\partial T_j$. It is assumed that the solution of (18) can be represented by a uniformly valid expansion having the form,

$$\xi_n(\tau, \epsilon) = \xi_{n0}(T_0, T_1, \ldots) + \epsilon \xi_{n1}(T_0, T_1, \ldots) + \cdots \quad (19a)$$

$$\eta_n(\tau, \epsilon) = \eta_{n0}(T_0, T_1, \ldots) + \epsilon \eta_{n1}(T_0, T_1, \ldots) + \cdots \quad (19b)$$

Due to the complexity of the problem, the expansion is carried out to the order of ϵ^2. Therefore, only T_0 and T_1 variables are needed. Substituting (19) into (18) and equating coefficients of like power of ϵ yields the following equations:

Order 1

$$D_0 \xi_{n0} - \omega_n \eta_{no} = 0, \quad (20a)$$

$$D_0 \eta_{n0} + \omega_n \xi_{no} = 0, \quad n=1, 2, \ldots, I \quad (20b)$$

Order ϵ

$$D_0 \xi_{n0} - \omega_n \eta_{no} = -D_1 \xi_{n0} - \alpha \sum_{m=1}^{I} (c_{nm}^{11} \xi_{m0} + c_{nm}^{12} \eta_{m0}) - \sum_{j=1}^{J} \sum_{m=1}^{I} (g_{jnm}^{11} \xi_{m0} + g_{jnm}^{12} \eta_{m0}) cos\, j\tilde{\Omega}T_0 - \sum_{j=1}^{J} \sum_{m=1}^{I} (h_{jnm}^{11} \xi_{m0} + h_{jnm}^{12} \eta_{m0}) sin\, j\tilde{\Omega}T_0], \quad (21a)$$

$$D_0 \eta_{n0} + \omega_n \xi_{no} = -D_1 \eta_{n0} - \alpha \sum_{m=1}^{I} (c_{nm}^{21} \xi_{m0} + c_{nm}^{22} \eta_{m0}) - \sum_{j=1}^{J} \sum_{m=1}^{I} (g_{jnm}^{21} \xi_{m0} + g_{jnm}^{22} \eta_{m0}) cos\, j\tilde{\Omega}T_0 - \sum_{j=1}^{J} \sum_{m=1}^{I} (h_{jnm}^{21} \xi_{m0} + h_{jnm}^{22} \eta_{m0}) sin\, j\tilde{\Omega}T_0], \quad n=1, 2, \ldots, I \quad (21b)$$

where $\tilde{c} = \epsilon\alpha$ is assumed to have the damping terms appear in the same order as the first excitation terms.

The general solution of (20) is found to be

$$\xi_{n0} = A_n(T_1) e^{i\omega_n T_0} + c.c., \quad (22a)$$

$$\eta_{n0} = iA_n(T_1) e^{i\omega_n T_0} + c.c., \quad n=1, 2, \ldots, I \quad (22b)$$

where c.c. denotes the complex conjugate of the preceding term, and A_n are undetermined functions of T_1. Substituting the above equation into (21) and making use of Euler's identity for $\cos j\tilde{\Omega}T_0$ and $\sin j\tilde{\Omega}T_0$ yields,

$$D_0 \xi_{n1} - \omega_n \eta_{n1} = -(D_1 A_n) e^{i\omega_n T_0} - \alpha \sum_{m=1}^{I} (c_{nm}^{11} + i c_{nm}^{12}) A_m e^{i\omega_m T_0} - \frac{1}{2} \sum_{j=1}^{J} \sum_{m=1}^{I} (g_{jnm}^{11} + i g_{jnm}^{12}) A_m (e^{i(\omega_m + j\tilde{\Omega})T_0} + e^{i(\omega_m - j\tilde{\Omega})T_0}) - \frac{1}{2} \sum_{j=1}^{J} \sum_{m=1}^{I} (-i h_{jnm}^{11} + h_{jnm}^{12}) A_m (e^{i(\omega_m + j\tilde{\Omega})T_0} + e^{i(\omega_m - j\tilde{\Omega})T_0}), \quad (23a)$$

$$D_0 \eta_{n0} + \omega_n \xi_{no} = -i(D_1 A_n) e^{i\omega_n T_0} - \alpha \sum_{m=1}^{I} (c_{nm}^{21} + i\, c_{nm}^{22}) A_m e^{i\omega_m T_0} - \frac{1}{2} \sum_{j=1}^{J} \sum_{m=1}^{I} (g_{jnm}^{21} + i g_{jnm}^{22}) A_m (e^{i(\omega_m + j\tilde{\Omega})T_0} + e^{i(\omega_m - j\tilde{\Omega})T_0}) - \frac{1}{2} \sum_{j=1}^{J} \sum_{m=1}^{I} (-i h_{jnm}^{21} + h_{jnm}^{22}) A_m (e^{i(\omega_m + j\tilde{\Omega})T_0} + e^{i(\omega_m - j\tilde{\Omega})T_0}), \quad n=1, 2, \ldots, I \quad (23b)$$

To this order of approximation, there exist three possible frequency combinations of $j\tilde{\Omega}$, ω_m and ω_n. All three possible cases will be investigated separately as follows.

A. The Case of $j\tilde{\Omega}$ Away from $\omega_p \pm \omega_q$.

When the excitation frequency multiples $j\tilde{\Omega}$ are away from the sum or the difference of any two natural frequencies of the system ω_p and ω_q, one deals with a nonresonant case. In this case, the secular terms – terms with $e^{i\omega_n T_0}$ – must be eliminated from (23), which results in

$$A_n = a_n e^{-\frac{\alpha}{2}(c_{nn}^{11} + c_{nn}^{22})T_1}, \quad n=1, 2, \ldots, I \quad (24)$$

where a_n are arbitrary functions of T_2. Note that the above equation is obtained as a result of $c_{nn}^{12} = c_{nn}^{21}$ since the matrix [C] is symmetric. The amplitude A_n will decay with time, and hence the system is always stable if $(c_{nn}^{11} + c_{nn}^{22})$ is positive,

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which is always true because the matrix [C] is always positive definite in this work.

B. The Case of $j\tilde{\Omega}$ Near $\omega_p + \omega_q$.

When the excitation frequency multiples $j\tilde{\Omega}$ are close to the sum of two natural frequencies of the system ω_p and ω_q, a combination resonance of the summed-type exists between the pth and the qth modes. In this case, elimination of the secular terms from (23) yields the expressions for the transition curves, which separate the stable regions from unstable ones,

$$j\tilde{\Omega} = \omega_p + \omega_q \pm \frac{\epsilon}{2}\sqrt{(\frac{\Lambda_{jpq}\bar{\Lambda}_{jqp}}{c_p c_q} - \alpha^2)(c_p + c_q)^2}, \quad (25)$$

where

$$c_p = c_{pp}^{11} + c_{pp}^{22},\ c_q = c_{qq}^{11} + c_{qq}^{22}, \quad (26a)$$

$$\Lambda_{jpq} = \frac{1}{2}(h_{jpq}^{21} - ih_{jpq}^{22} + ig_{jpq}^{21} + g_{jpq}^{22} + ih_{jpq}^{11} + h_{jpq}^{12} - g_{jpq}^{11} + ig_{jpq}^{12}), \quad (26b)$$

$$\bar{\Lambda}_{jqp} = \frac{1}{2}(h_{jqp}^{21} + ih_{jqp}^{22} - ig_{jqp}^{21} + g_{jqp}^{22} - ih_{jqp}^{11} + h_{jqp}^{12} - g_{jqp}^{11} - ig_{jqp}^{12}). \quad (26c)$$

C. The Case of $j\tilde{\Omega}$ Near $\omega_p - \omega_q$.

When the excitation frequency multiples $j\tilde{\Omega}$ are close to the difference of two natural frequencies of the system ω_p and ω_q, a combination resonance of the difference-type exists between the pth and the qth modes. In this case, elimination of the secular terms from (23) yields the expressions for the transition curves,

$$j\tilde{\Omega} = \omega_p - \omega_q \pm \frac{\epsilon}{2}\sqrt{(\frac{\Lambda_{jpq}\Lambda_{jqp}}{c_p c_q} - \alpha^2)(c_p + c_q)^2}, \quad (27)$$

Here c_p and c_q are the same as those in the summed-type case, but Λ_{jpq} and Λ_{jpq} have different expressions,

$$\Lambda_{jpq} = \frac{1}{2}(h_{jpq}^{21} + ih_{jpq}^{22} + ig_{jpq}^{21} - g_{jpq}^{22} + ih_{jpq}^{11} - h_{jpq}^{12} - g_{jpq}^{11} - ig_{jpq}^{12}), \quad (28a)$$

$$\Lambda_{jqp} = \frac{1}{2}(-h_{jqp}^{21} - ih_{jqp}^{22} + ig_{jqp}^{21} - g_{jqp}^{22} - ih_{jqp}^{11} + h_{jqp}^{12} - g_{jqp}^{11} - ig_{jqp}^{12}). \quad (28b)$$

IV. Numerical Results and Discussions

As an example of the application of the general solutions obtained in this work, the periodic excitation is taken as

$$f(y,t) = f_0(y)(1 + \epsilon\cos\Omega t), \quad (29)$$

where ϵ is the small parameter in the perturbation analysis.

When the internal damping of a viscoelastic plate c=0, the plate becomes elastic. Fig. 2 shows the stability boundaries of an axially moving elastic square plate subjected to uniformly distributed in-plane edge excitation. In this case $f_0(y) = f_0$ is a constant. Figs. 2 (a)–(c) show the stability boundaries corresponding to the lowest three main resonances at twice the 1st, 2nd and 3rd natural frequencies of the axially moving plate, respectively; Figs. 2 (d)–(f) show those corresponding to the lowest three summed-type combination resonances. No stability boundaries corresponding to difference-type combination resonances are found. Each pair of stability boundaries sprouts from the abscissa at points where the frequency of the in-plane edge excitation is equal to twice the natural frequencies or sum of two natural frequencies of the axially moving plate. Every stability boundary is a line because the problem is linear, and only the first order approximation is carried out. Unstable regions lie within a pair of stability boundaries. It is observed that unstable regions become smaller with higher main resonances. Also, the unstable region at summed-type combination resonance between the 1st and 3rd natural modes is larger than unstable regions at the other two summed-type combination resonances because the 1st and 3rd modes are symmetric in the axial direction, while the 2nd mode is antisymmetric in the axial direction. Generally, unstable regions corresponding to main resonances are much larger than those corresponding to summed-type combination resonances

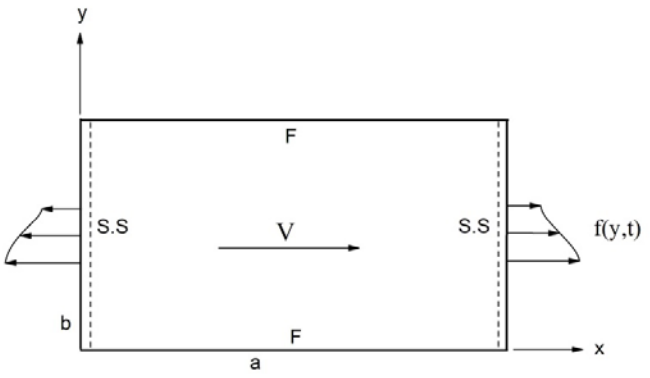
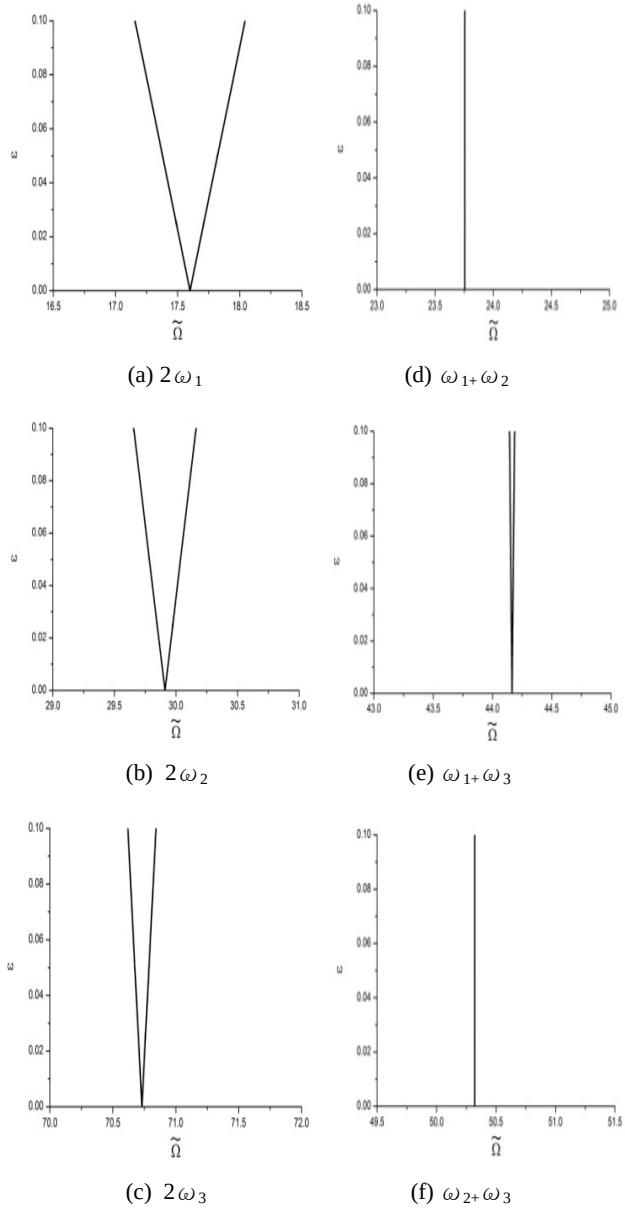


Fig. 1 Schematic plot of an axially moving viscoelastic plate under non-uniform in-plane edge excitation

When a rectangular plate is subjected to linearly distributed in-plane edge excitations, the in-plane stress resultants equal the corresponding linearly distributed in-plane edge excitation. In that case, one can deal only with the out-of-plane equation, (2c), and the in-plane inertia of the plate is neglected. Fig. 3 presents the stability boundaries of an axially moving elastic square plate subjected to uniformly distributed in-plane edge excitation with and without consideration of the in-plane inertia of the plate. In this figure, the solid lines represent the stability boundaries with consideration of the in-plane inertia of the plate; while the dot lines represent the stability boundaries without consideration of the in-plane inertia of the plate. It is found that the sizes of unstable regions with consideration of the in-plane inertia of the plate are almost the same as those without consideration of the in-plane inertia of the plate. The reason for this is attributed to the fact that the lowest few natural frequencies of the transverse vibration are much smaller than the natural frequencies of the in-plane vibration for a thin plate. But the positions of unstable regions may be shifted a little bit due to the effect of the in-plane inertia of the plate.

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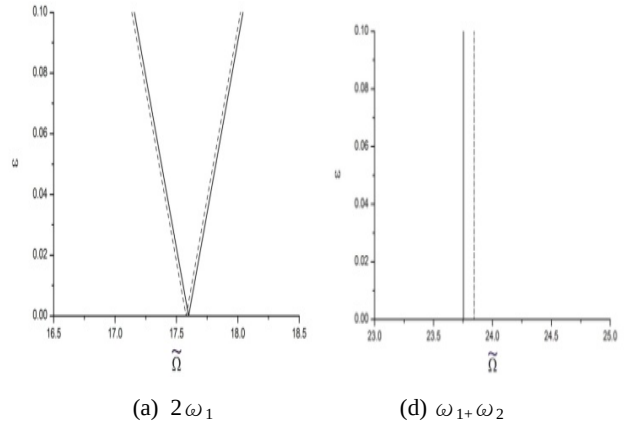


(a) $2\omega_1$ (d) $\omega_{1+}\omega_2$

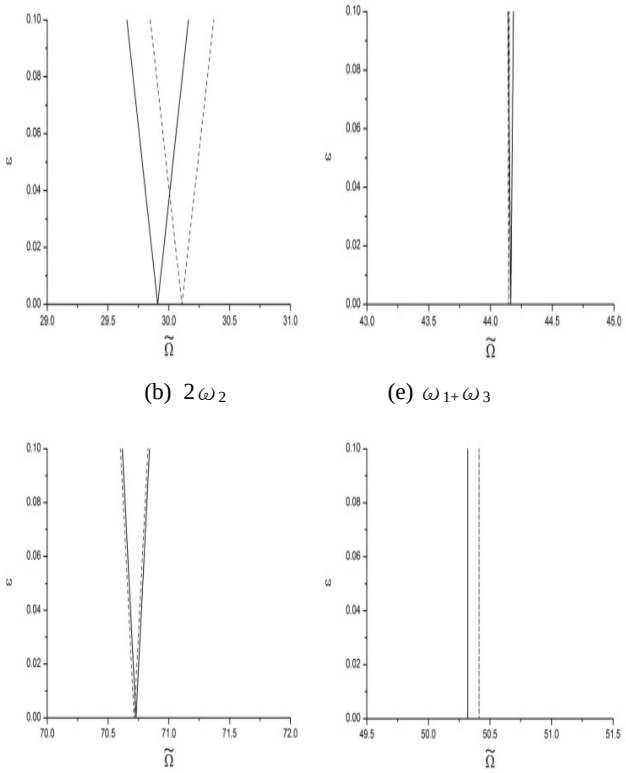
(b) $2\omega_2$ (e) $\omega_{1+}\omega_3$

(c) $2\omega_3$ (f) $\omega_{2+}\omega_3$

Fig. 2 Stability boundaries of an axially moving elastic square plate subjected to uniformly distributed in-plane edge excitation



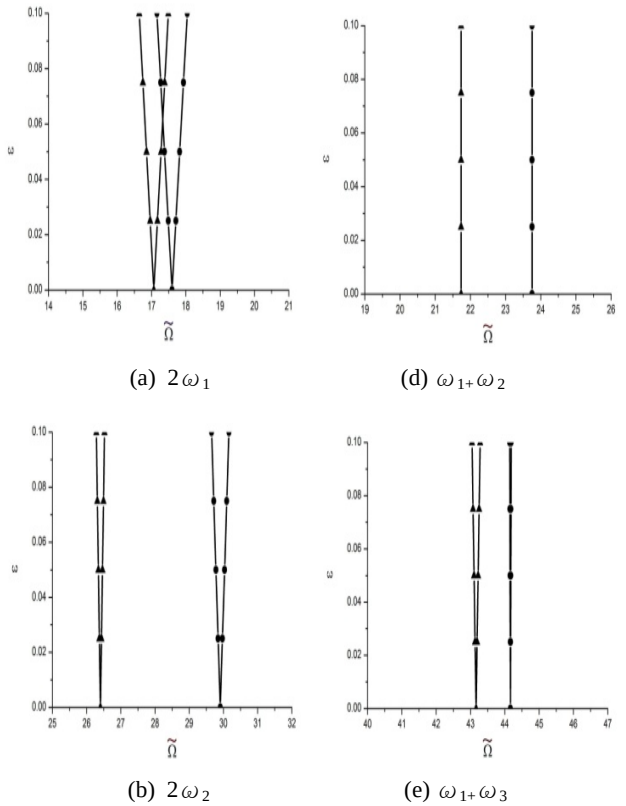
(a) $2\omega_1$ (d) $\omega_{1+}\omega_2$



(b) $2\omega_2$ (e) $\omega_{1+}\omega_3$

(c) $2\omega_3$ (f) $\omega_{2+}\omega_3$

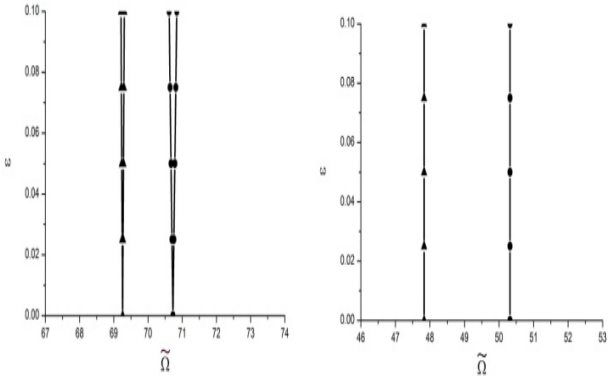
Fig. 3 Stability boundaries of an axially moving elastic square plate subjected to uniformly distributed in-plane edge excitation. Solid line: with consideration of the in-plane inertia of the plate; dot line: without consideration of the in-plane inertia of the plate



(a) $2\omega_1$ (d) $\omega_{1+}\omega_2$

(b) $2\omega_2$ (e) $\omega_{1+}\omega_3$

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(c) $2\omega_3$ (f) $\omega_{2+}\omega_3$

Fig. 4 Stability boundaries of an axially moving elastic square plate subjected to in-plane edge excitation. Circle: uniformly distributed; triangular: partially distributed on the middle half of the edge

Fig. 4 illustrated the stability boundaries of an axially moving elastic square plate subjected to in-plane edge excitation either uniformly distributed or partially distributed on the middle half of the edge. The intensity of the in-plane edge excitation partially distributed on the middle half of the edge is twice that of the uniformly distributed in-plane edge excitation such that the total magnitude of the in-plane edge excitation applied in both cases is the same. One finds that unstable regions corresponding to partially distributed edge excitation become smaller generally and shift toward the lower frequency range.

The effect of the aspect ratio on the stability boundaries of an axially moving elastic rectangular plate subjected to in-plane excitation partially distributed on the middle half of the edge is depicted in Fig. 5. Fig. 5 (a) shows the stability boundaries of a square plate; Figs. 5 (b) and (c) show the stability boundaries of a rectangular plate. The figure reveals that as the aspect ratio increases, all unstable regions reduce in size, shift toward the lower frequency range and get closer.

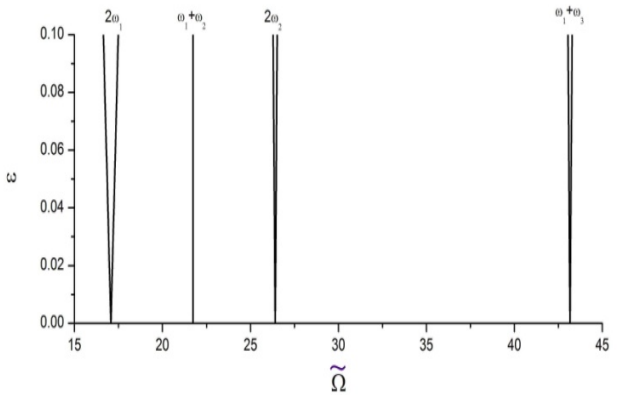
The effect of the internal damping on the stability boundaries of an axially moving viscoelastic square plate subjected to uniformly distributed in-plane edge excitation is presented in Fig. 6. Comparing with Fig. 2, it is discovered that all unstable regions appear at the same location but shrink upwards, and some small unstable regions even disappear due to the presence of internal damping. Therefore, the effect of internal damping is favorable to the stability of the axially moving plate.

V. Conclusions

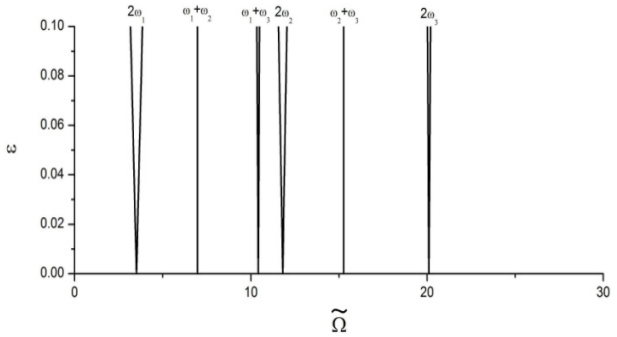
Dynamic stability of an axially moving plate subjected to non-uniform in-plane edge excitation has been studied analytically in this work. Due to complexity of the problem, only the first order approximation was presented, and a simple harmonic excitation was considered to provide numerical illustration of the general solution sought. However, solutions for more general periodic excitations can be easily generated for the first few approximations.

From the above numerical results, the following conclusions can be drawn:

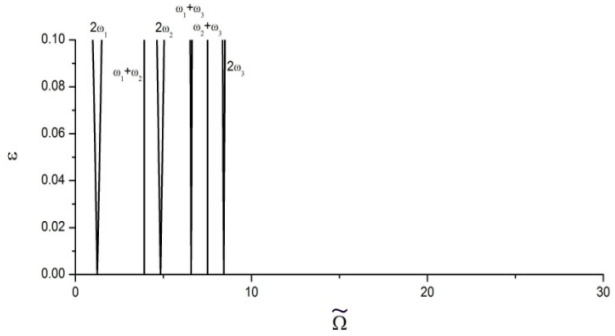
1) For an axially moving plate subjected to nonuniform in-plane edge excitation, combination resonances of the difference-type do not exist, and unstable regions for summed-type combination resonances are general smaller than those for main resonances.
2) When a rectangular plate is subjected to linearly distributed in-plane edge excitation, the in-plane stress resultants equal the corresponding linearly distributed in-plane edge excitation. In that case, one can deal only with the out-of-plane equation, and the in-plane inertia of the plate is neglected if the plate is thin.
3) When an axially moving plate is subjected to symmetrically distributed in-plane edge excitation, unstable regions become smaller generally and shift toward the lower frequency range as the distribution width of the excitation becomes narrower.
4) As the aspect ratio increases, all unstable regions of an axially moving plate under in-plane edge excitation reduce in size, shift toward the lower frequency range and get closer.
5) The effect of the internal damping will make all unstable regions of an axially moving plate shrink upwards but still appear at the same location.



(a) $a/b = 1$

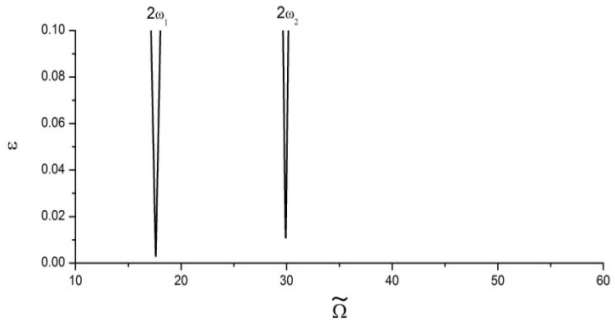


(b) $a/b = 2$

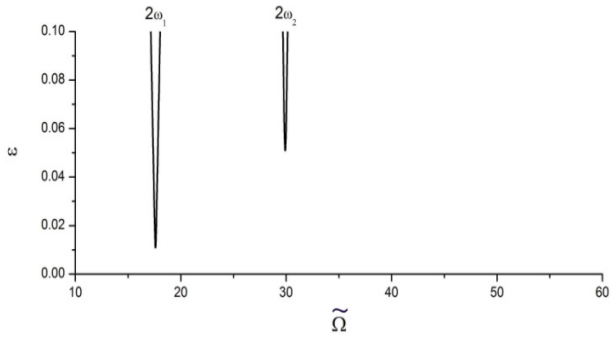


(c) $a/b = 3$

Fig. 5 Stability boundaries of an axially moving elastic rectangular plate subjected to in-plane excitation partially distributed on the middle half of the edge



(a) *c*=0.0001



(b) *c*=0.0005

Fig. 6 Stability boundaries of an axially moving viscoelastic square plate subjected to uniformly distributed in-plane edge excitation

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